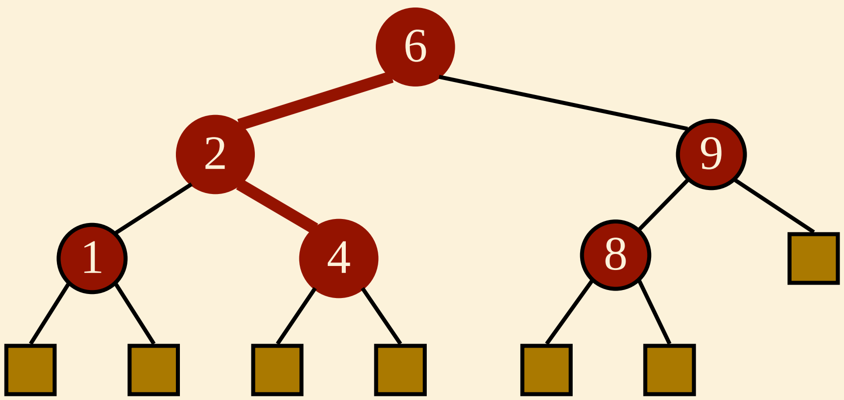
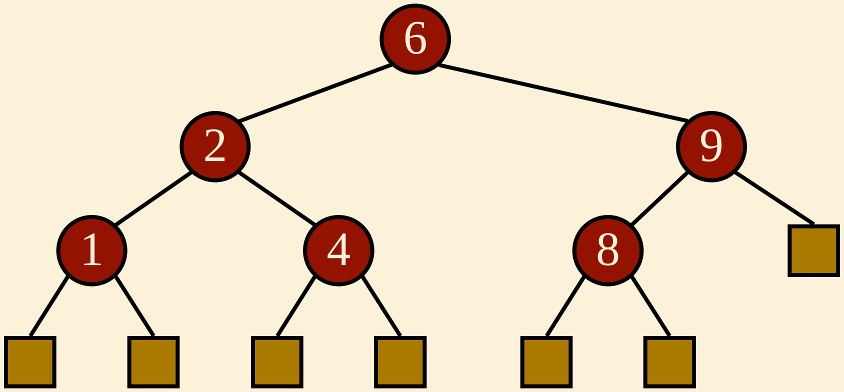
# Search Trees

Chapter 10



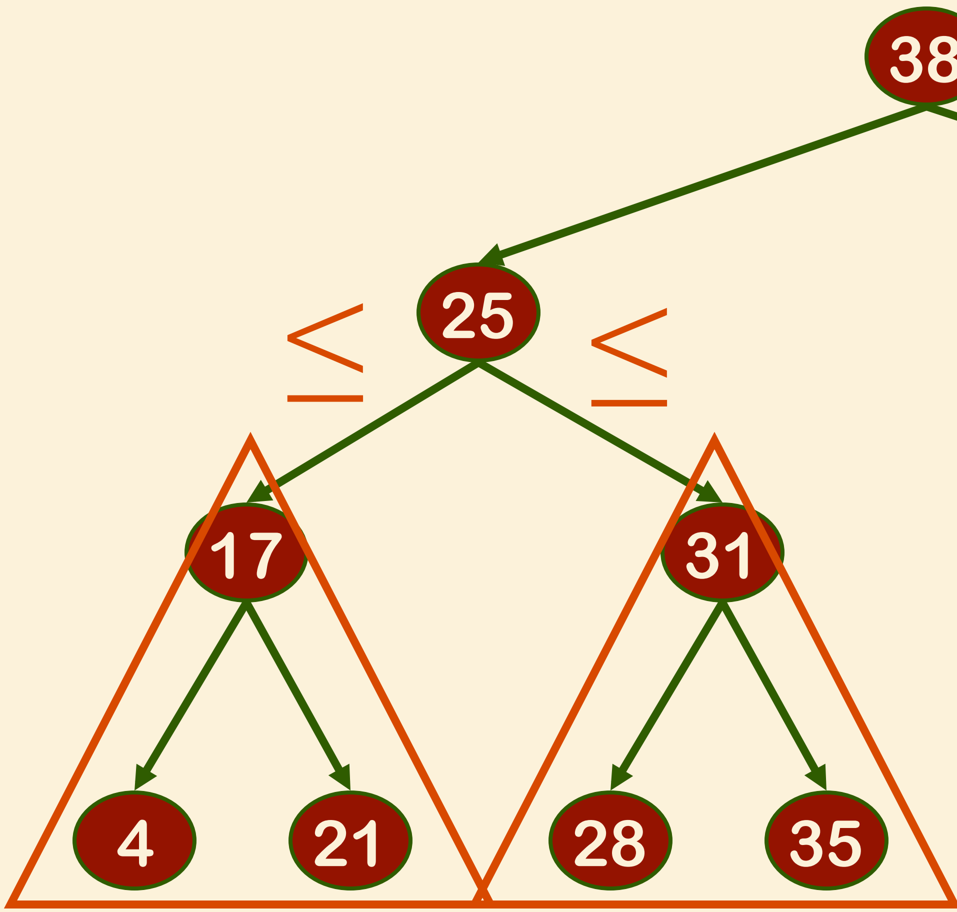
# Binary Search Trees

- A binary search tree is a binary tree storing key-value entries at its internal nodes and satisfying the following property:
  - Let ***u***, ***v***, and ***w*** be three nodes such that ***u*** is in the left subtree of ***v*** and ***w*** is in the right subtree of ***v***. We have ***key***(***u***) ≤ ***key***(***v***) ≤ ***key***(***w***)
- The textbook assumes that external nodes are 'placeholders': they do not store entries (makes algorithms a little simpler)
- An inorder traversal of a binary search trees visits the keys in increasing order
- Binary search trees are ideal for dictionaries with ordered keys.



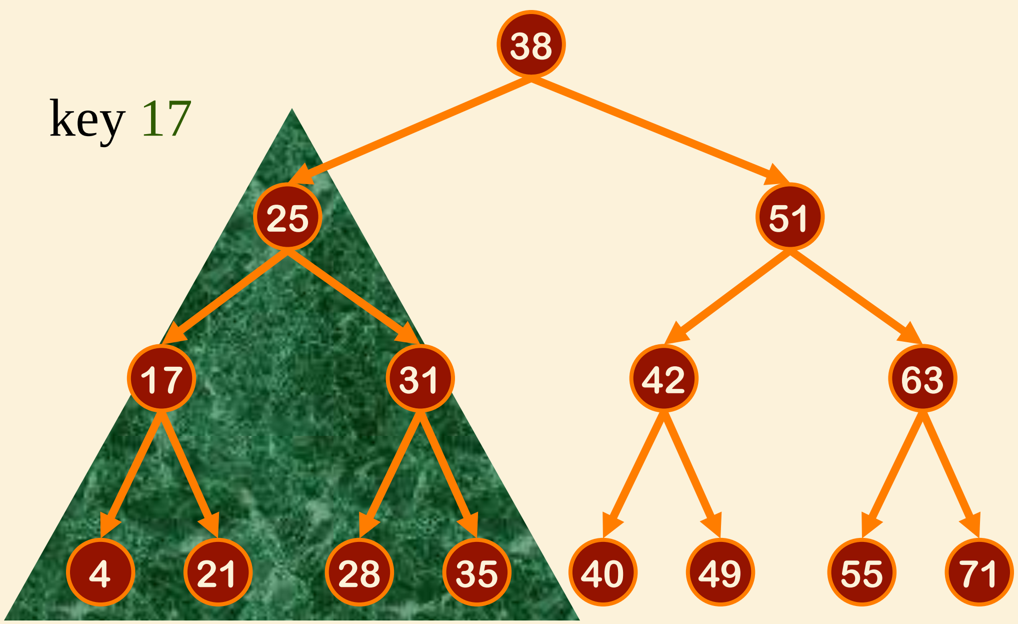
# Binary Search Tree

All nodes in left subtree ≤ Any node ≤ All nodes in right subtree



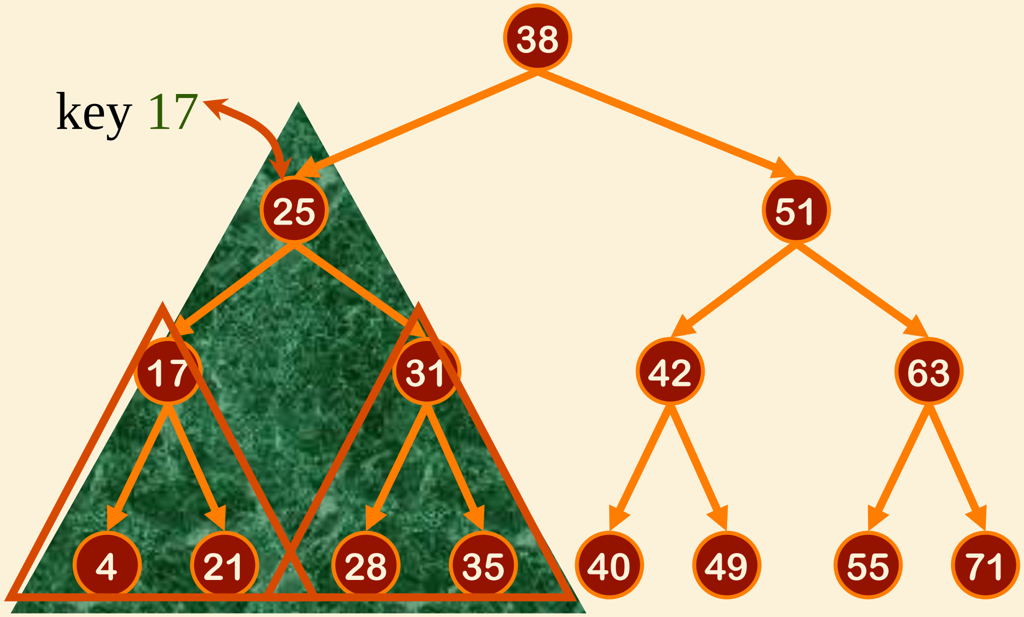
# Search: Loop Invariant

- Maintain a sub-tree.
- If the key is contained in the original tree, then the key is contained in the sub-tree.



# Search: Define Step

- Cut sub-tree in half.
- Determine which half the key would be in.
- Keep that half.




If key < root, then key is in left half.

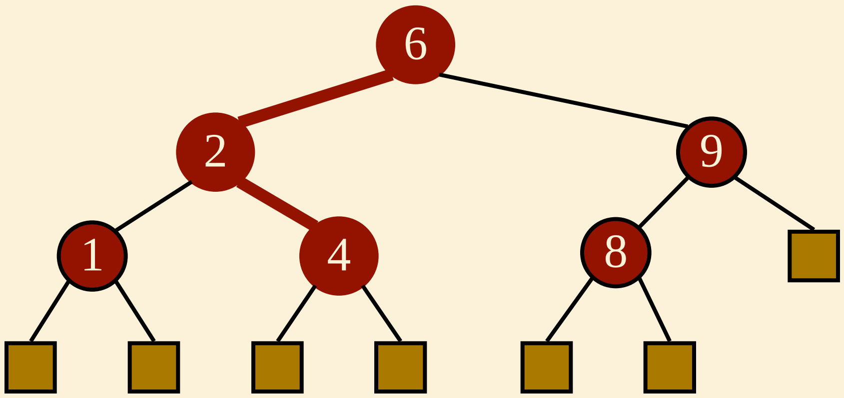
If key = root, then key is found

If key > root, then key is in right half.

# Search: Algorithm

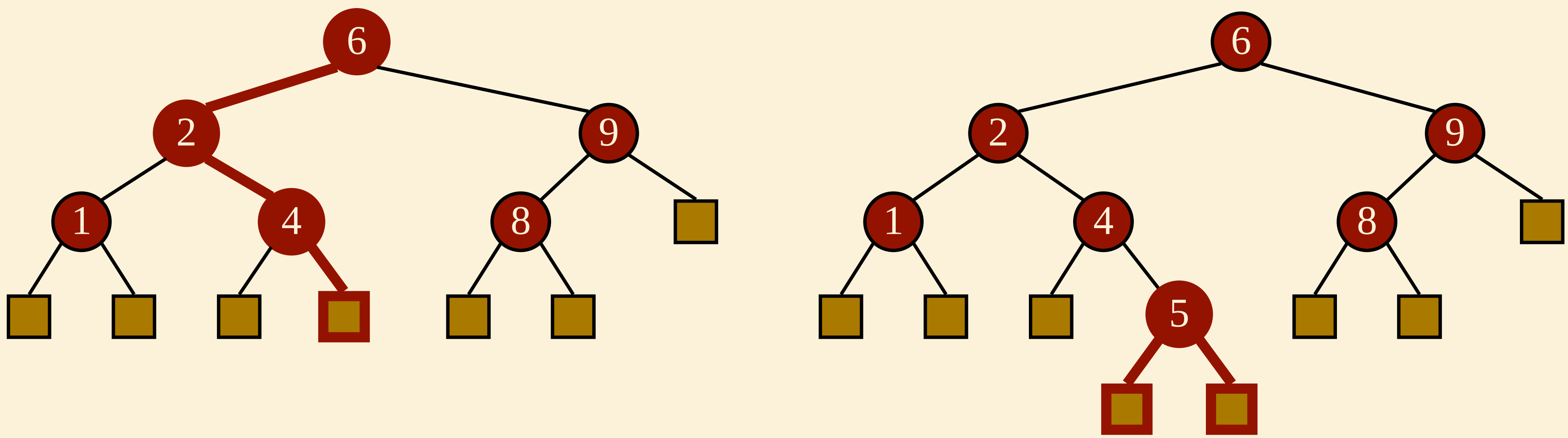
- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited depends on the outcome of the comparison of ***k*** with the key of the current node
- If we reach a leaf, the key is not found and return of an external node signals this.
- Example: find(4):
  - Call TreeSearch(4,root)

```
Algorithm TreeSearch(k, v)
    if T.isExternal (v)
        return v
    if k < key(v)
        return TreeSearch(k, T.left(v))
    else if k = key(v)
        return v
    else { k > key(v) }
        return TreeSearch(k, T.right(v))
```



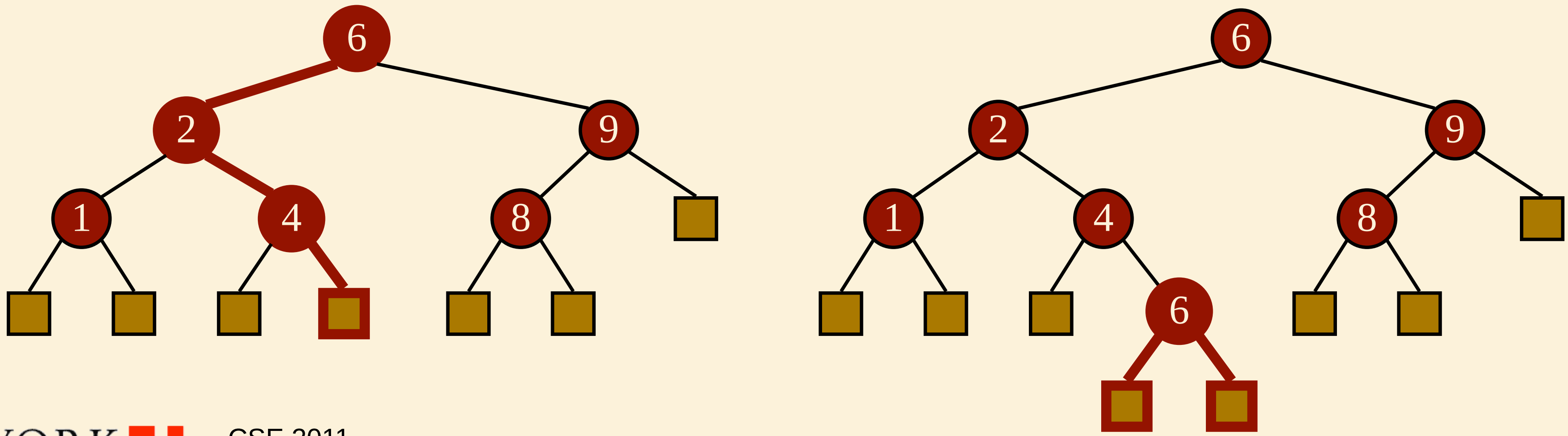
# Insertion

- To perform operation insert(k, o), we search for key **k** (using TreeSearch)
- Suppose **k** is not already in the tree, and let **w** be the leaf reached by the search
- We insert **k** at node **w** and expand **w** into an internal node
- Example: insert 5



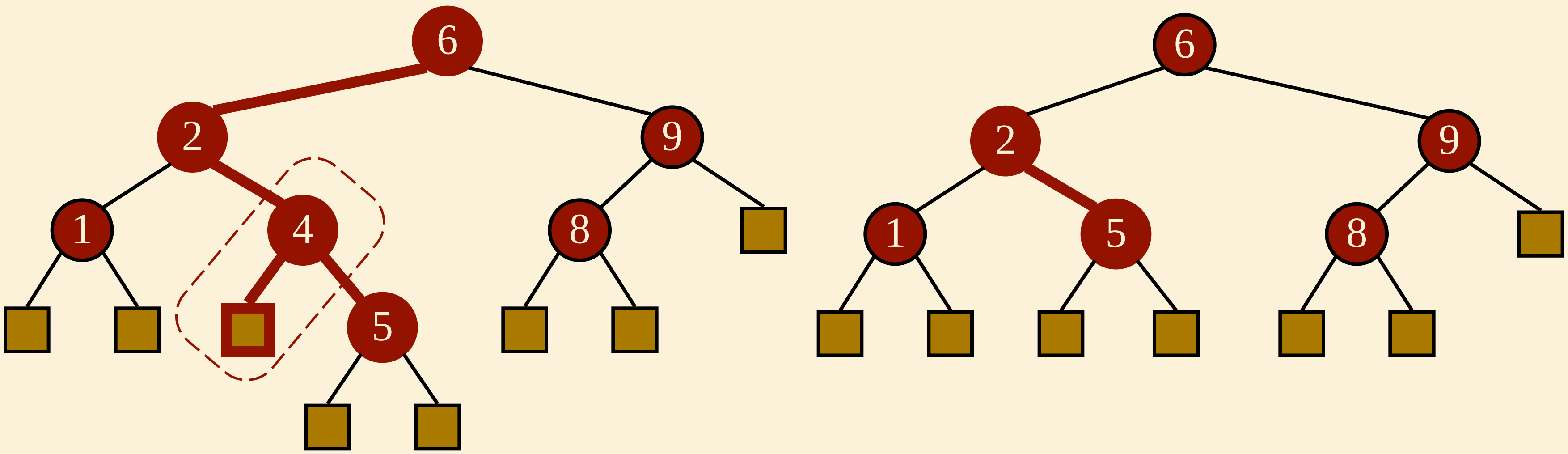
# Insertion

- Suppose **k is** already in the tree, at node **v**.
- We continue the downward search through **v**, and let **w** be the leaf reached by the search
- Note that it would be correct to go either left or right at **v**. We go left by convention.
- We insert **k** at node **w** and expand **w** into an internal node
- Example: insert 6



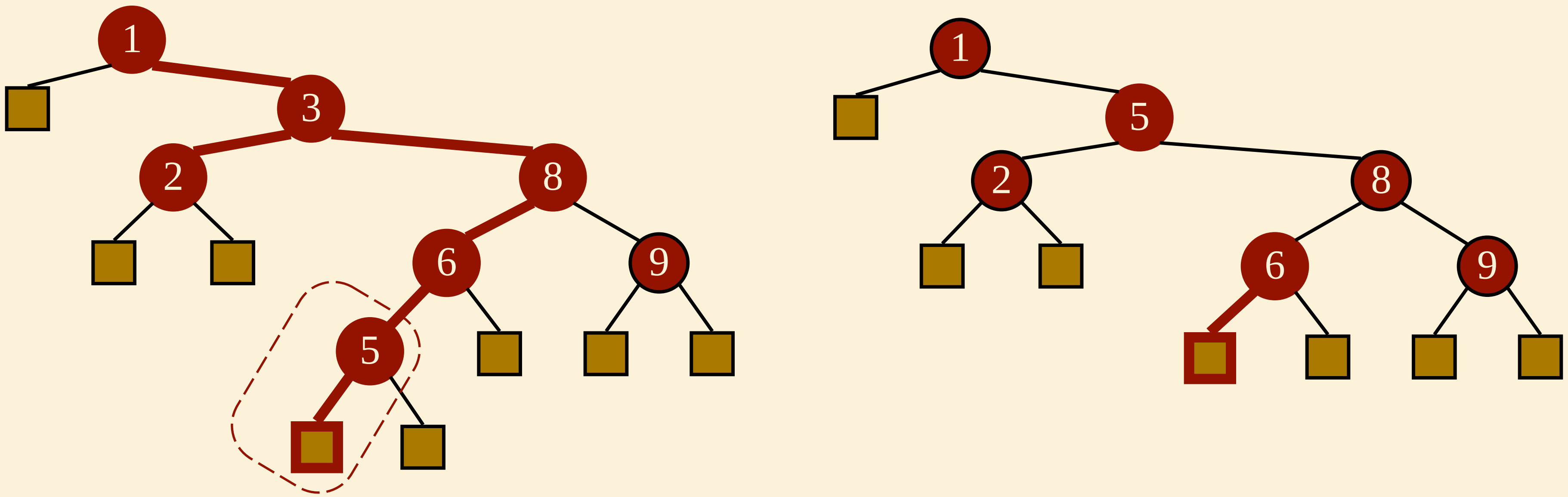
# Deletion

- To perform operation remove(***k***), we search for key ***k***
- Suppose key ***k*** is in the tree, and let ***v*** be the node storing ***k***
- If node ***v*** has a leaf child ***w***, we remove ***v*** and ***w*** from the tree with operation removeExternal(***w***), which removes ***w*** and its parent
- Example: remove 4



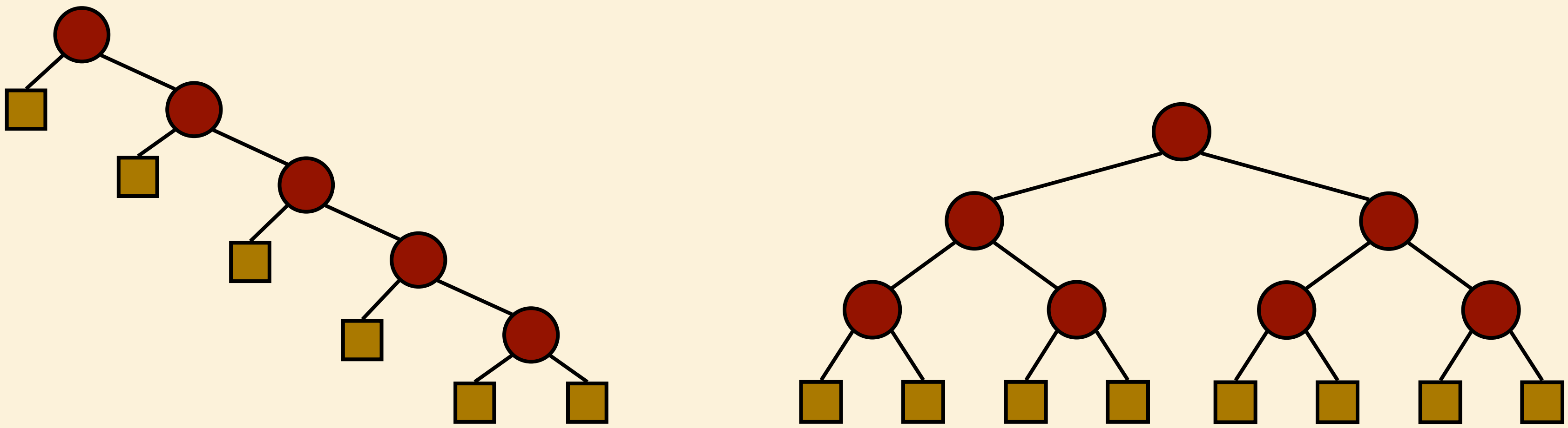
# Deletion (cont.)

- Now consider the case where the key ***k*** to be removed is stored at a node ***v*** whose children are both internal
  - we find the internal node ***w*** that follows ***v*** in an inorder traversal
  - we copy ***key***(***w***) into node ***v***
  - we remove node ***w*** and its left child ***z*** (which must be a leaf) by means of operation removeExternal(***z***)
- Example: remove 3

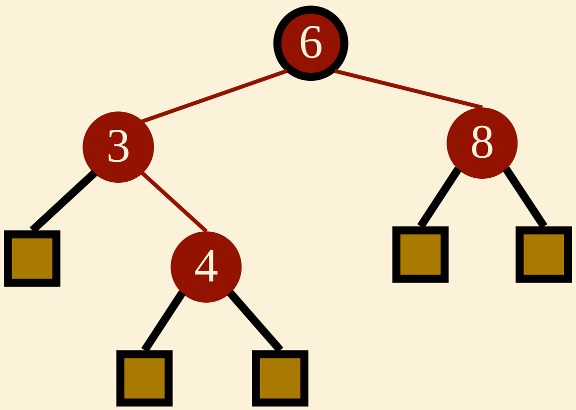


# Performance

- Consider a dictionary with $n$ items implemented by means of a binary search tree of height $h$
  - the space used is $O(n)$
  - methods find, insert and remove take $O(h)$ time
- The height $h$ is $O(n)$ in the worst case and $O(\log n)$ in the best case
- It is thus worthwhile to balance the tree (next topic)!



# AVL Trees

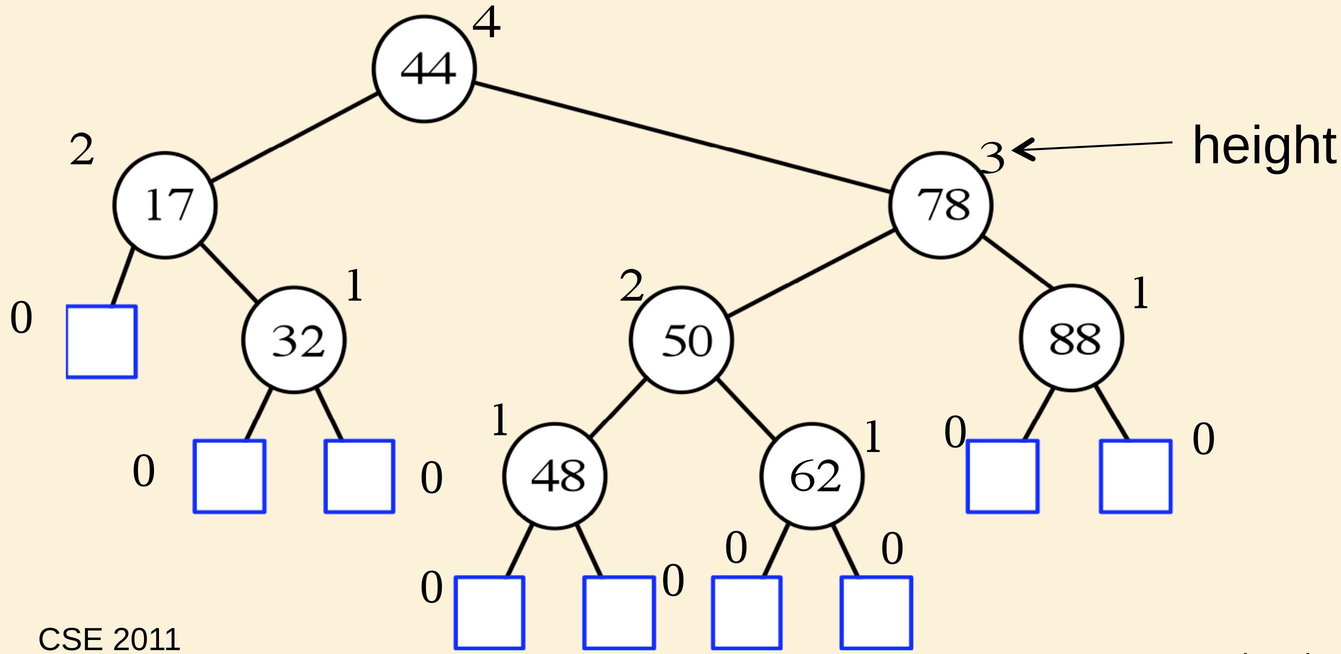


# AVL Trees

- The AVL tree is the first balanced binary search tree ever invented.
- It is named after its two inventors, G.M. Adelson-Velskii and E.M. Landis, who published it in their 1962 paper "An algorithm for the organization of information."

# AVL Trees

- **AVL trees are balanced.**
- An AVL Tree is a **binary search tree** in which the heights of siblings can differ by at most 1.

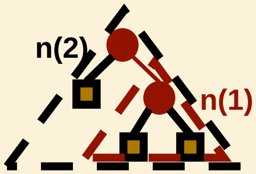


# Height of an AVL Tree

- **Claim**: The ***height*** of an AVL tree storing n keys is O(log n).

# Height of an AVL Tree

- **Proof**: We compute a lower bound **n(h)** on the number of internal nodes of an AVL tree of height h.
- Observe that n(1) = 1 and n(2) = 2

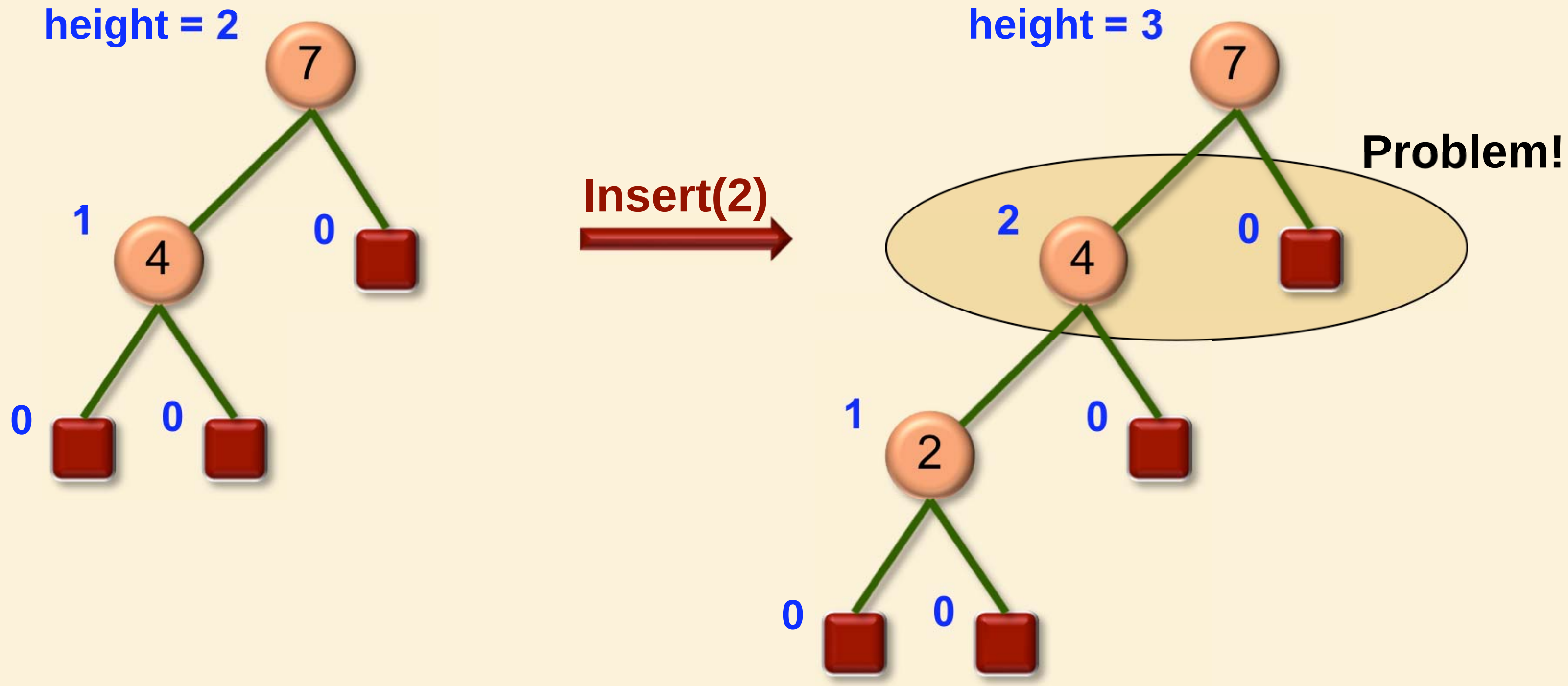



- For h > 2, a minimal AVL tree contains the root node, one AVL subtree of height h - 1 and another of height h - 2.
- That is, n(h) = 1 + n(h - 1) + n(h - 2)
- Knowing n(h - 1) > n(h - 2), we get n(h) > 2n(h - 2). So

  $n(h) > 2n(h - 2)$, $n(h) > 4n(h - 4)$, $n(h) > 8n(n - 6)$, … $> 2^i n(h - 2i)$

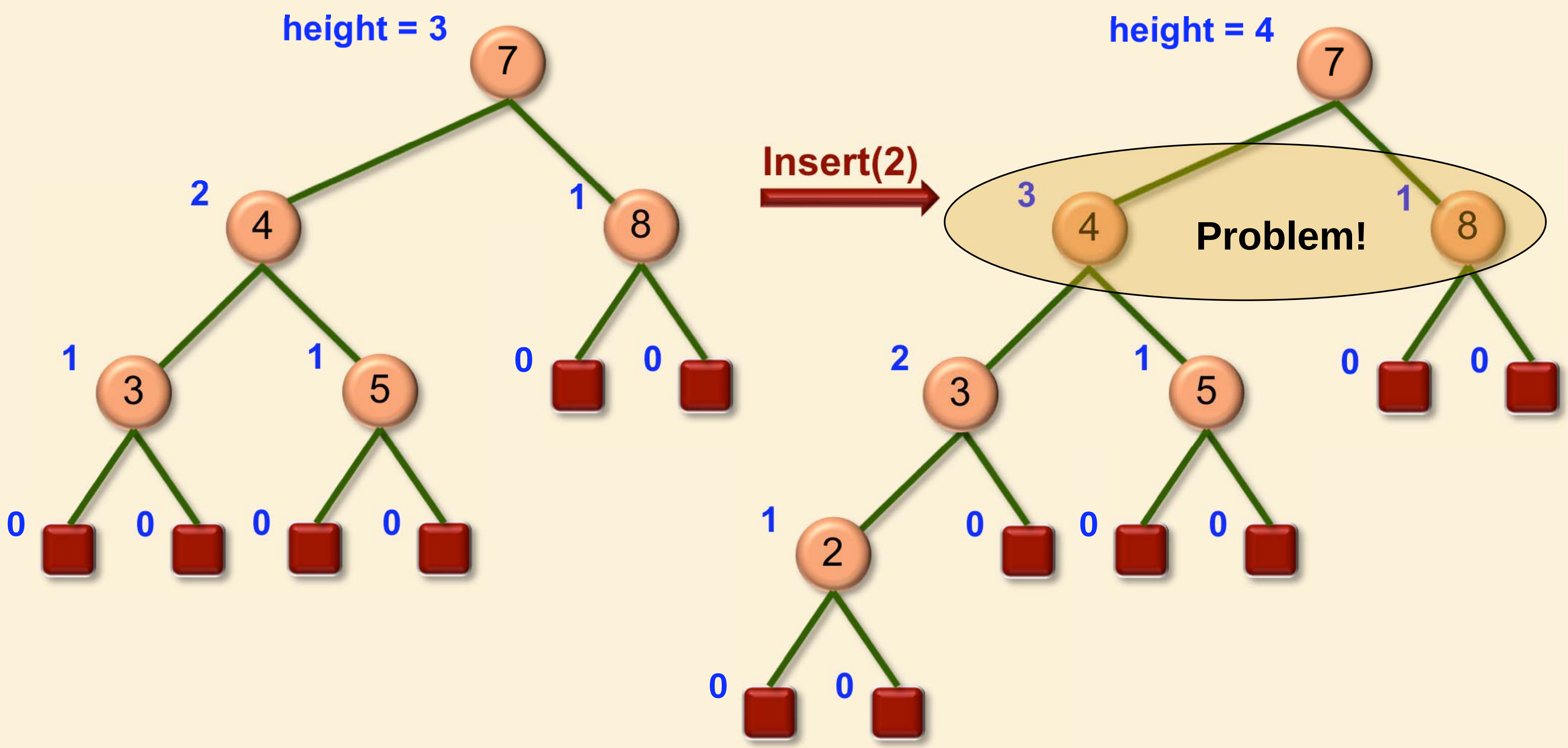
- If h is even, we let i = h/2-1, so that $n(h) > 2^{h/2-1} n(2) = 2^{h/2}$
- If h is odd, we let i = h/2-1/2, so that $n(h) > 2^{h/2-1/2} n(1) = 2^{h/2-1/2}$
- In either case, $n(h) > 2^{h/2-1}$
- Taking logarithms: h < 2log(n(h)) +2
- Thus the height of an AVL tree is O(log n)

# Insertion



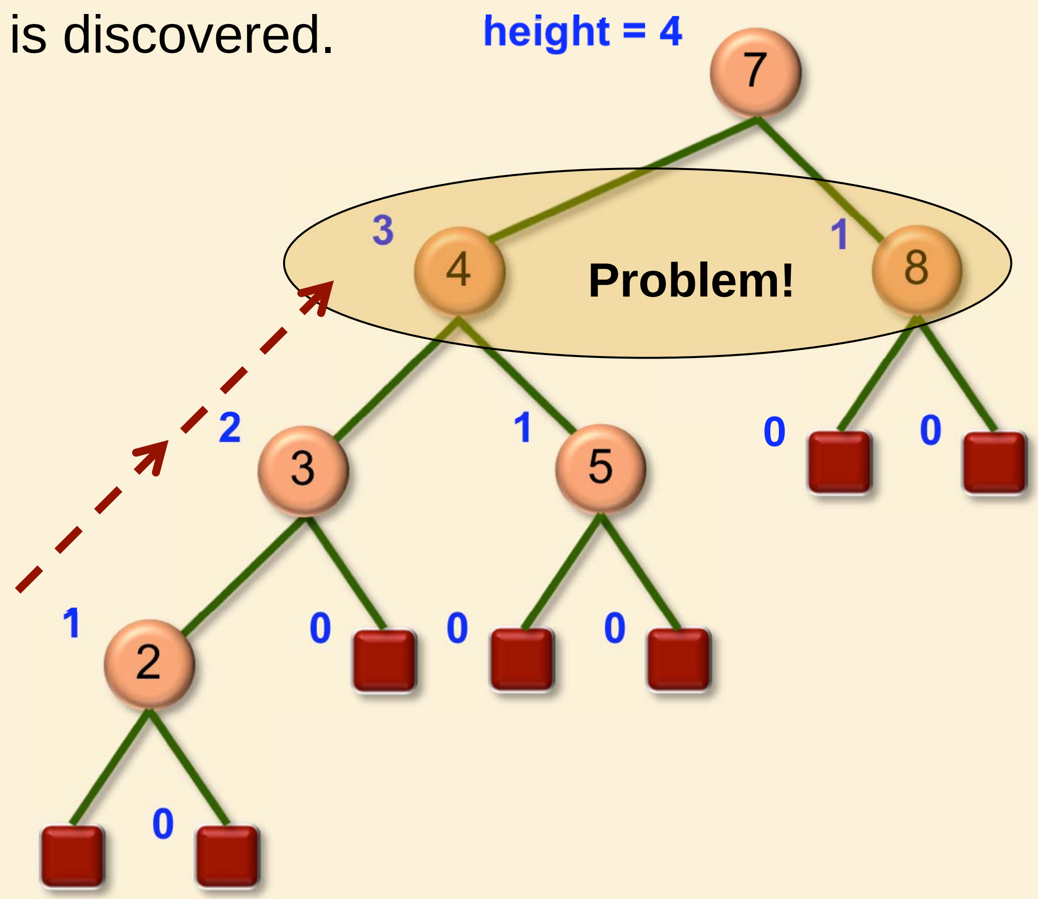
# Insertion

- Imbalance may occur at any ancestor of the inserted node.



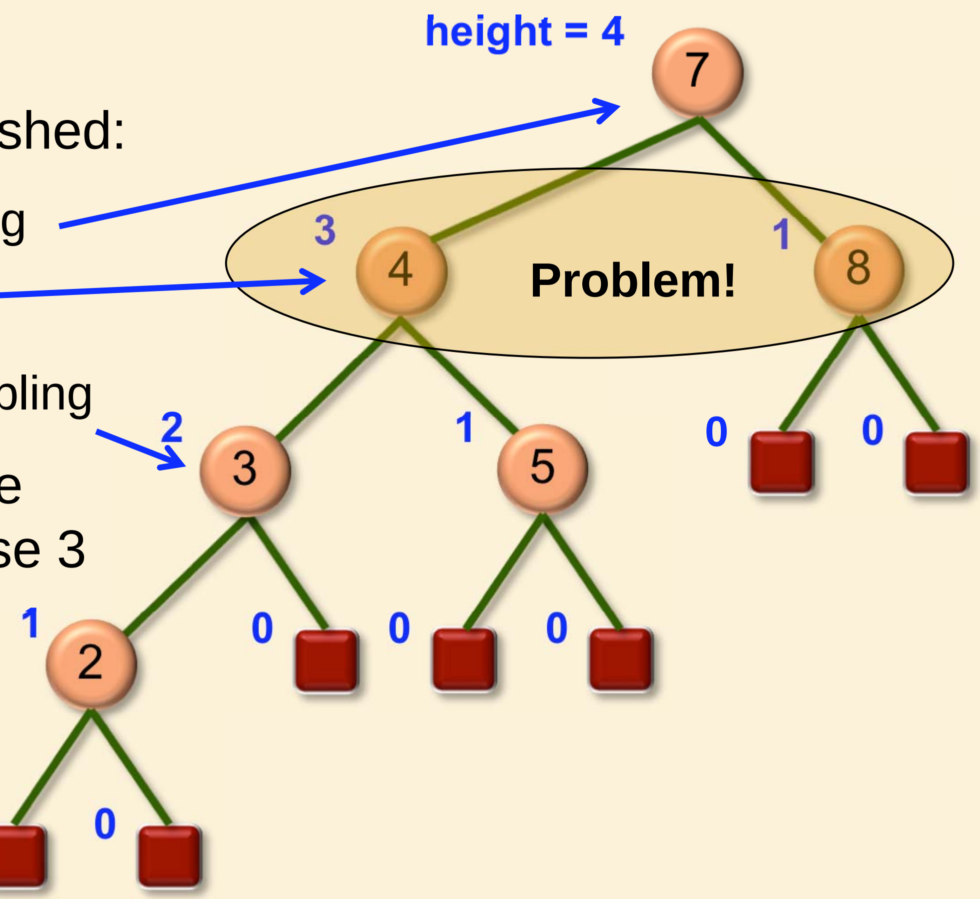
# Insertion: Rebalancing Strategy

- Step 1: Search
  - Starting at the inserted node, traverse toward the root until an imbalance is discovered.



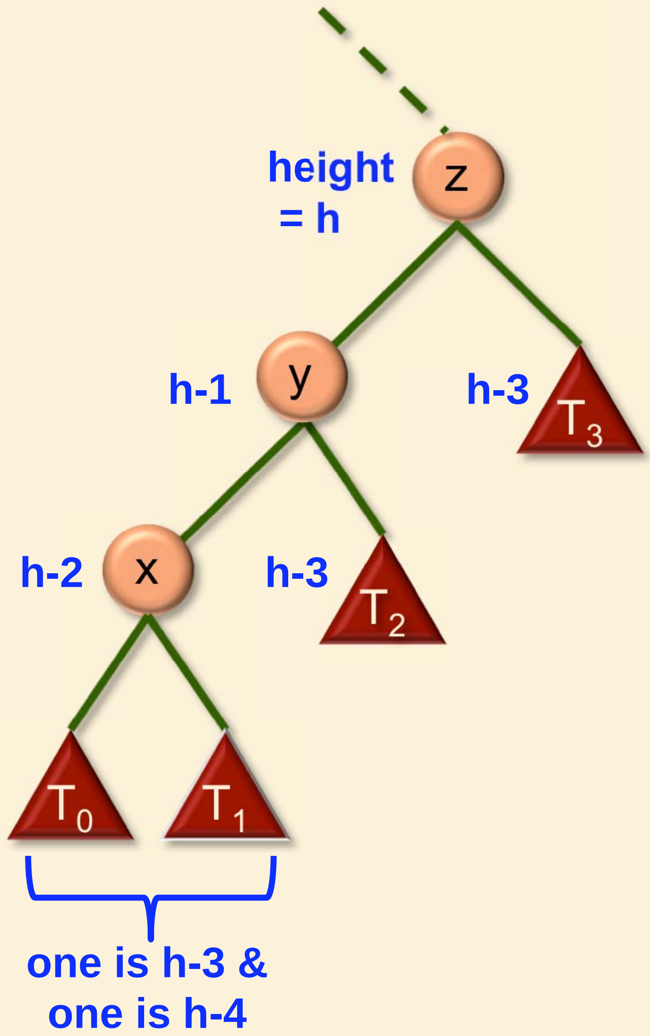
# Insertion: Rebalancing Strategy

- Step 2: Repair
  - The repair strategy is called **trinode restructuring**.
  - 3 nodes x, y and z are distinguished:
    - z = the parent of the high sibling
    - y = the high sibling
    - x = the high child of the high sibling
  - We can now think of the subtree rooted at z as consisting of these 3 nodes plus their 4 subtrees



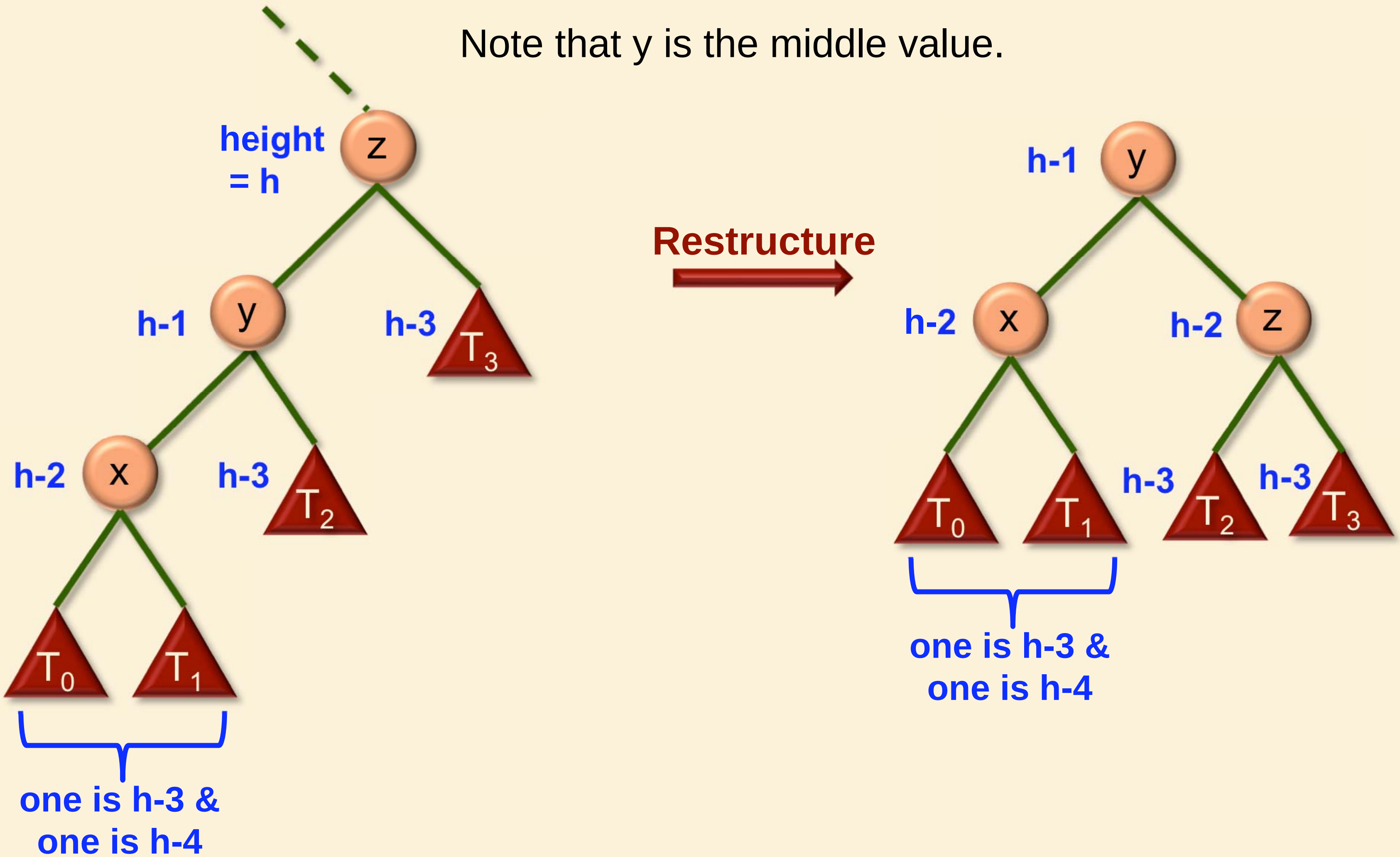
# Insertion: Rebalancing Strategy

- Step 2: Repair
  - The idea is to rearrange these 3 nodes so that the middle value becomes the root and the other two becomes its children.
  - Thus the linear **grandparent – parent – child** structure becomes a triangular **parent – two children** structure.
  - Note that **z** must be either bigger than both **x** and **y** or smaller than both **x** and **y**.
  - Thus either **x** or **y** is made the root of this subtree, and **z** is lowered by 1.
  - Then the subtrees $T_0 - T_3$ are attached at the appropriate places.
  - Since the heights of subtrees $T_0 - T_3$ differ by at most 1, the resulting tree is balanced.



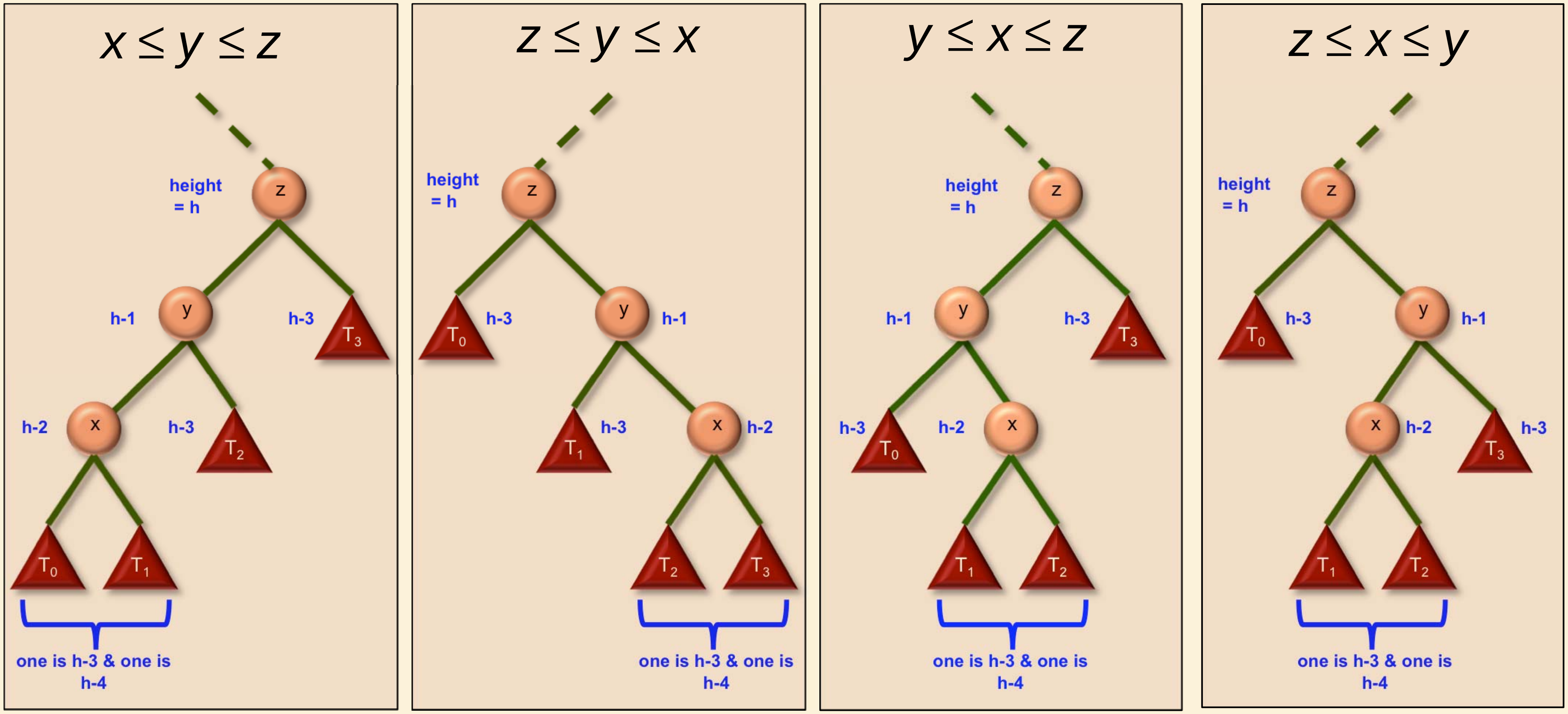
# Insertion: Trinode Restructuring Example

Note that y is the middle value.



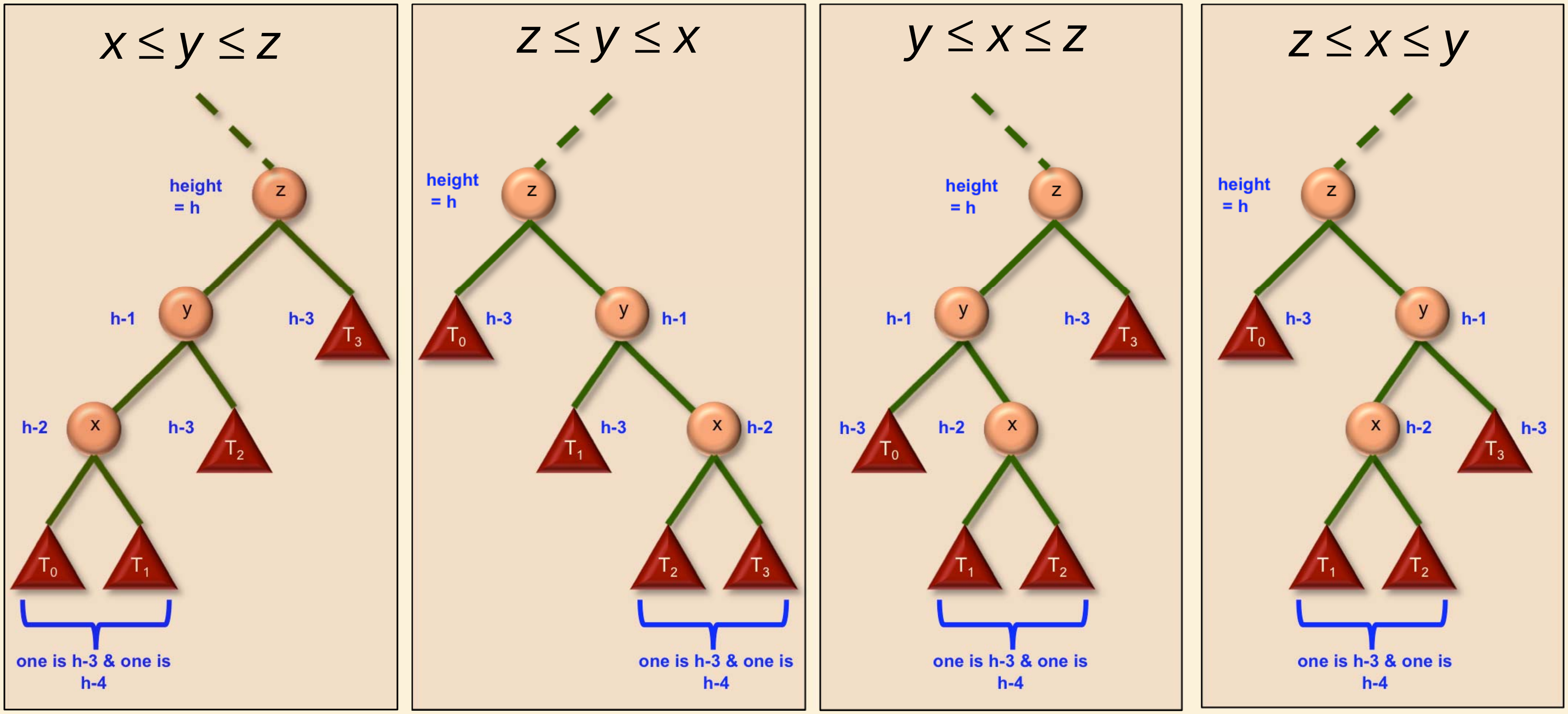
# Insertion: Trinode Restructuring - 4 Cases

- There are 4 different possible relationships between the three nodes x, y and z before restructuring:



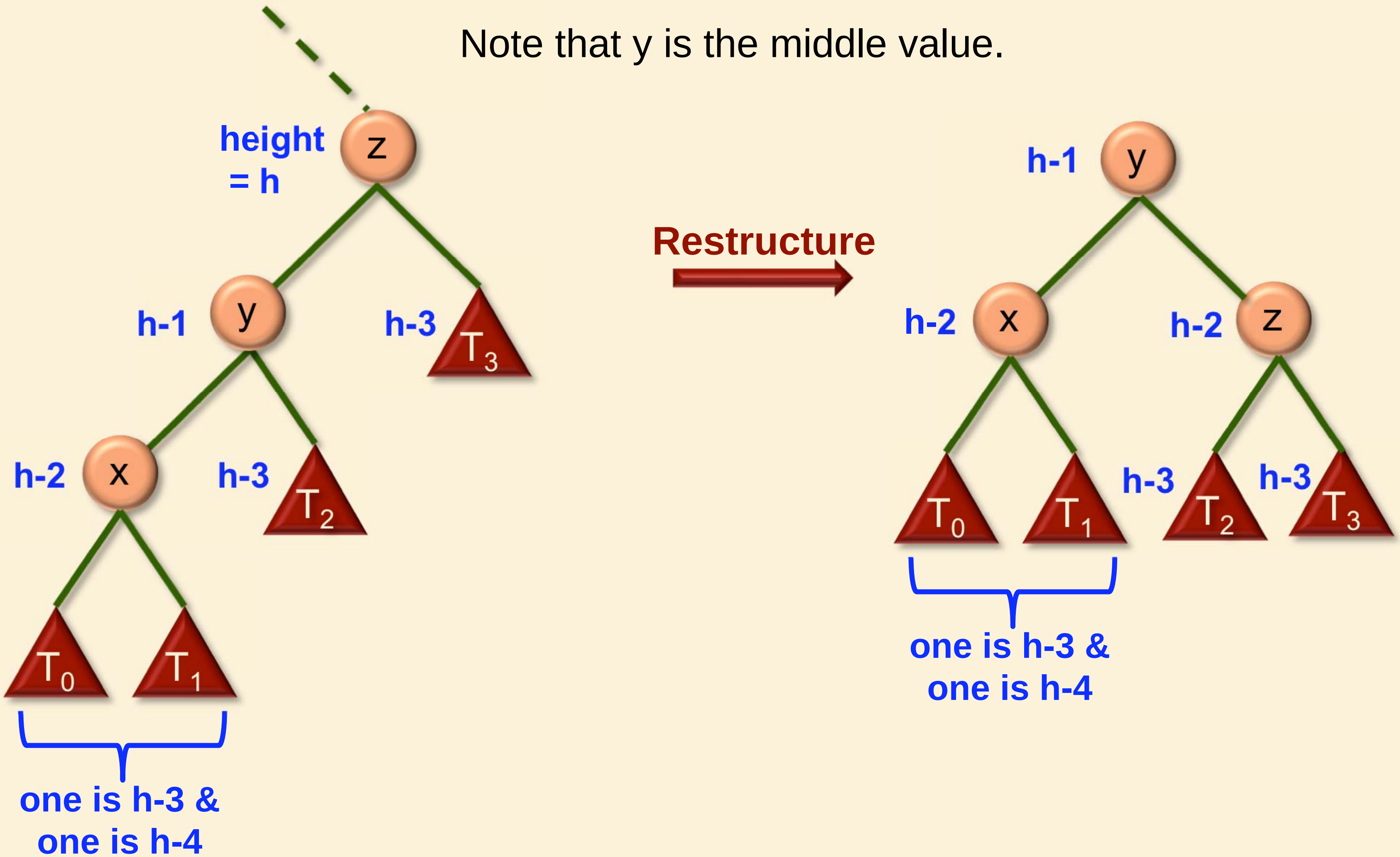
# Insertion: Trinode Restructuring - 4 Cases

- This leads to 4 different solutions, all based on the same principle.

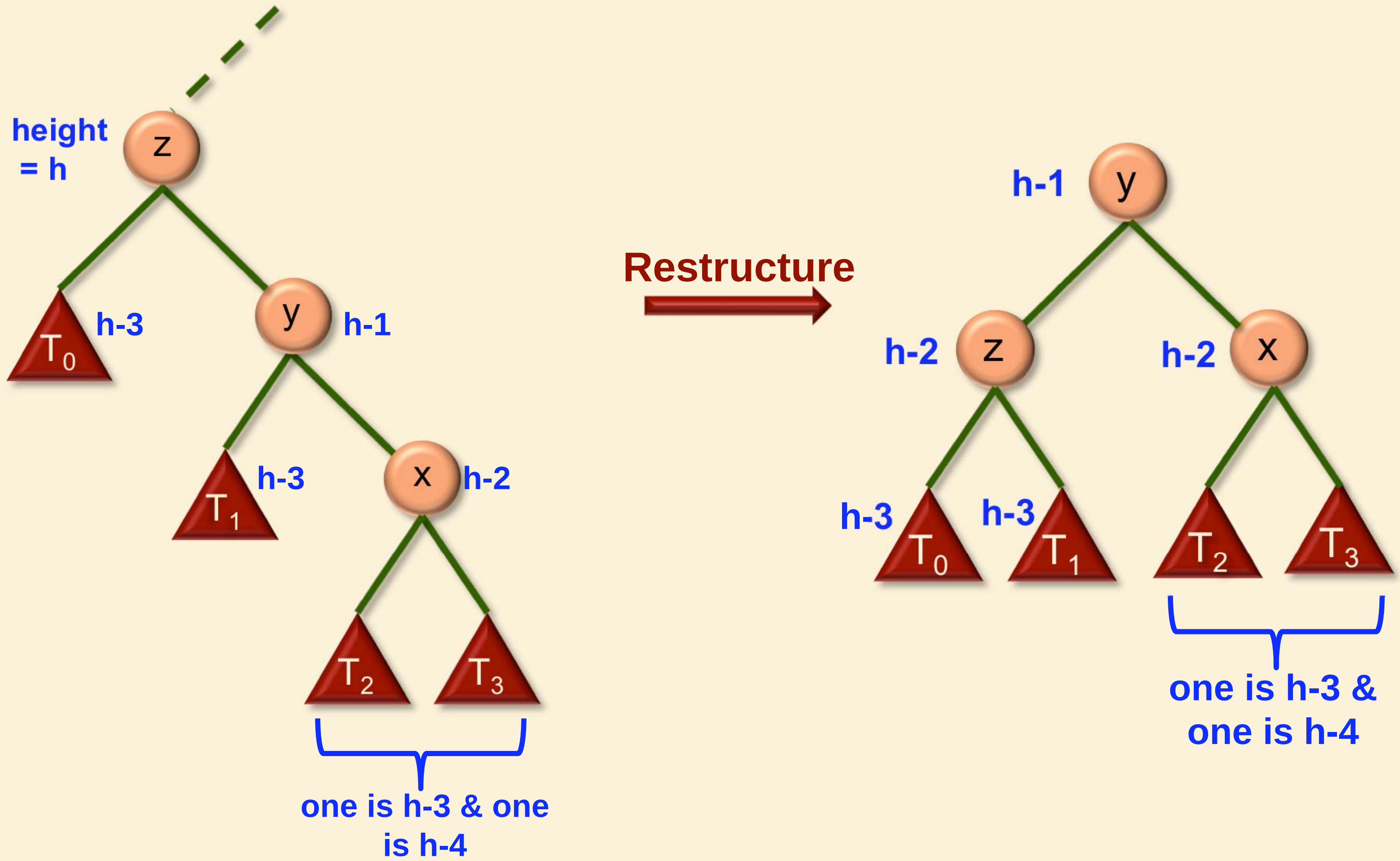


# Insertion: Trinode Restructuring - Case 1

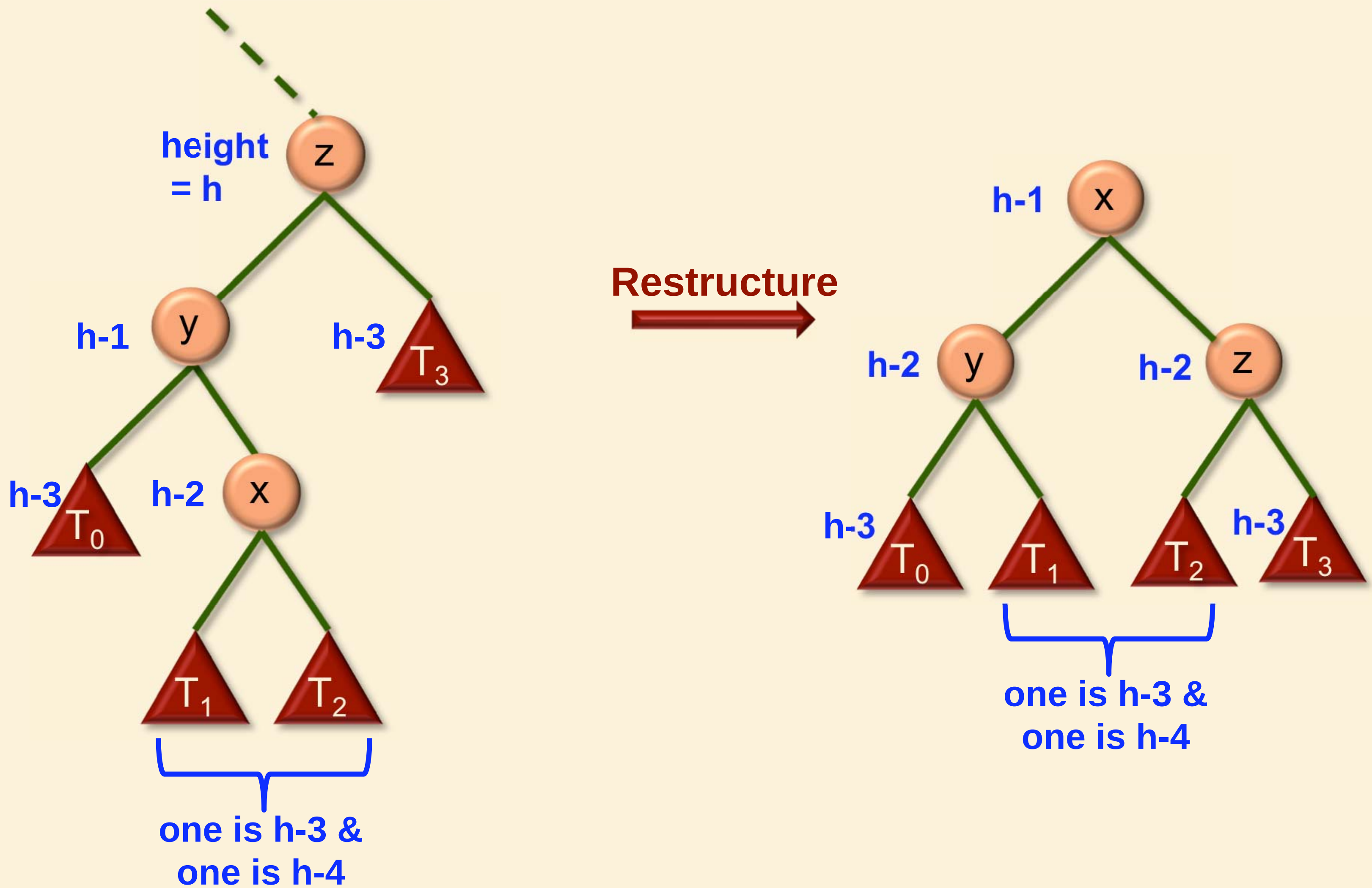
Note that y is the middle value.



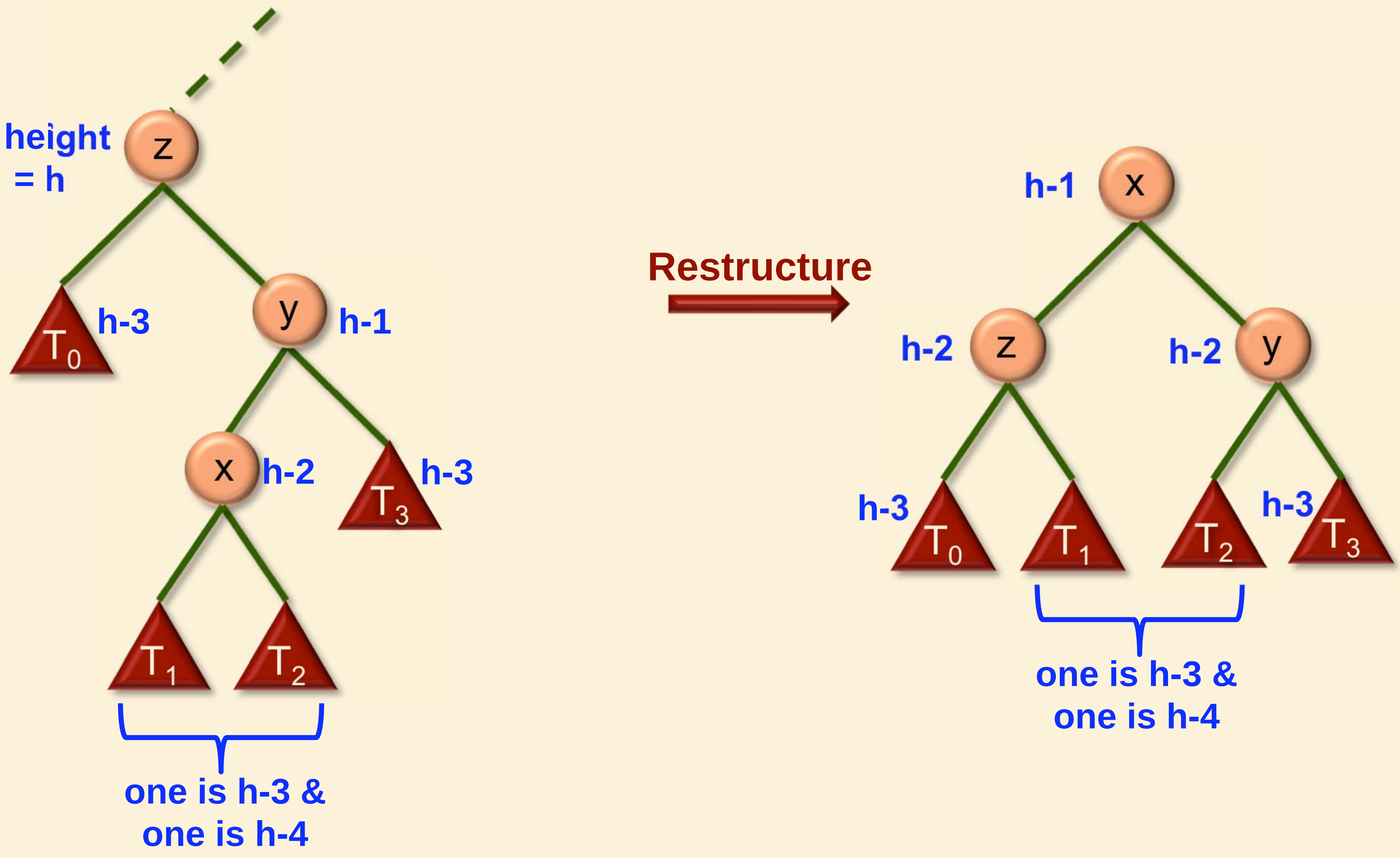
# Insertion: Trinode Restructuring - Case 2



# Insertion: Trinode Restructuring - Case 3

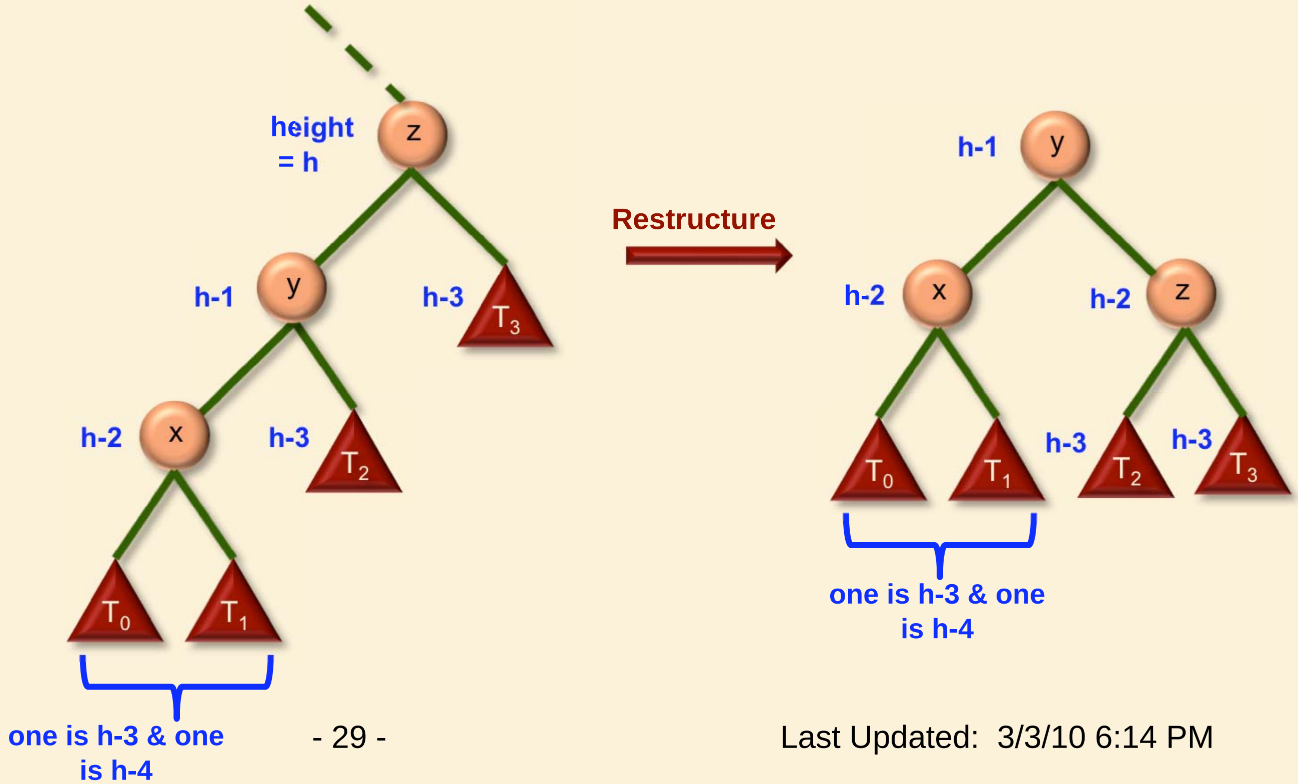


# Insertion: Trinode Restructuring - Case 4



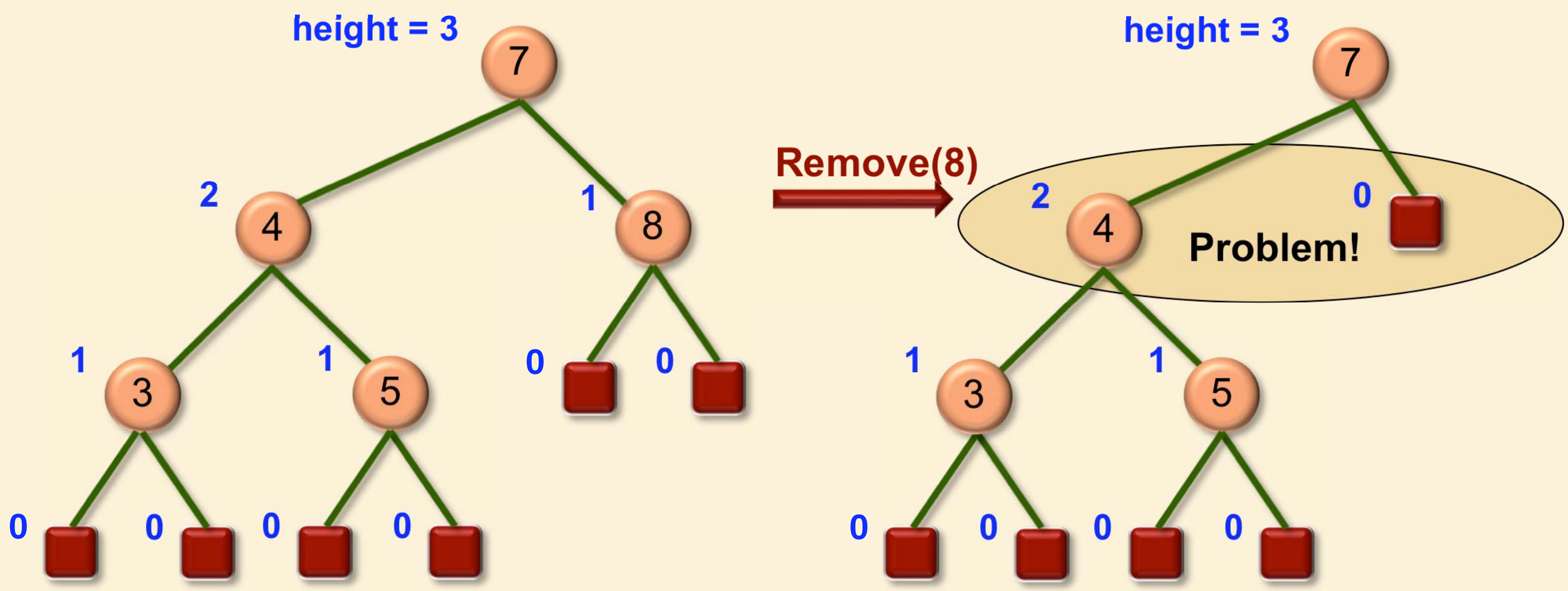
# Insertion: Trinode Restructuring - The Whole Tree

- Do we have to repeat this process further up the tree?
- No!
  - The tree was balanced before the insertion.
  - Insertion raised the height of the subtree by 1.
  - Rebalancing lowered the height of the subtree by 1.
  - Thus the whole tree is still balanced.



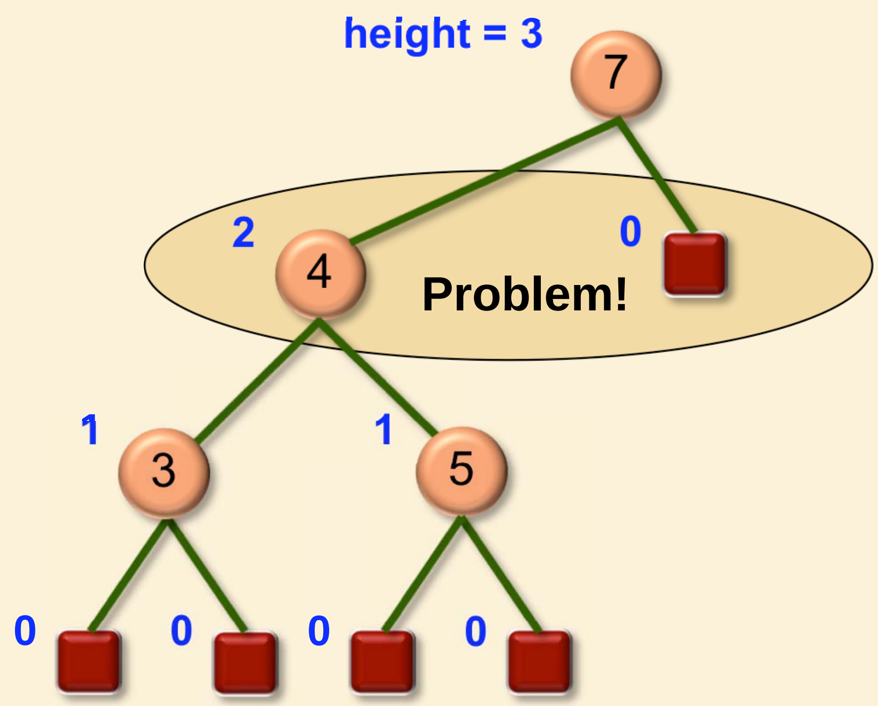
# Removal

- Imbalance may occur at an ancestor of the removed node.



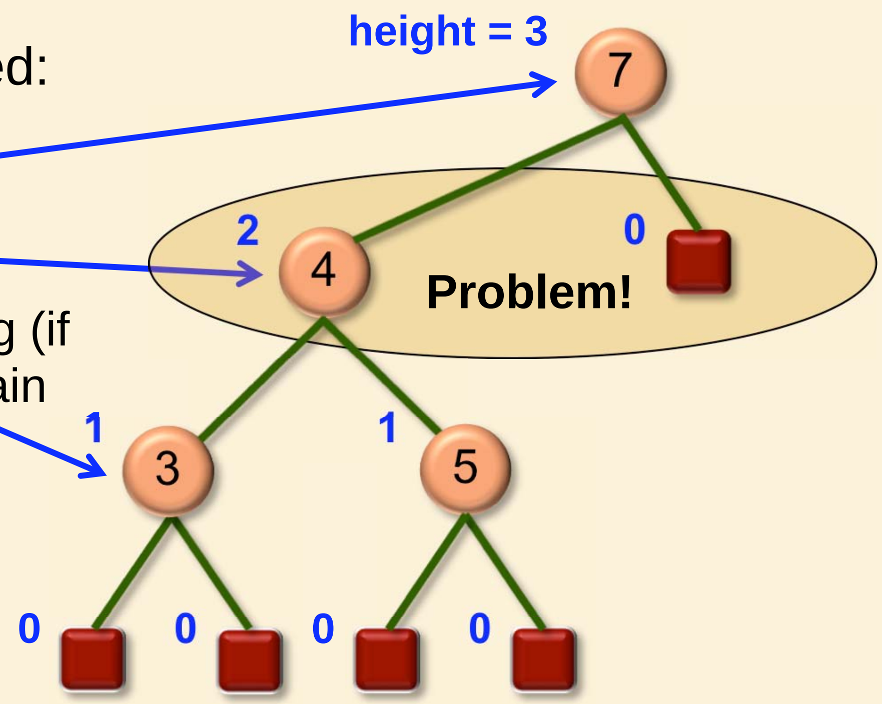
# Removal: Rebalancing Strategy

- Step 1: Search
  - Starting at the location of the removed node, traverse toward the root until an imbalance is discovered.



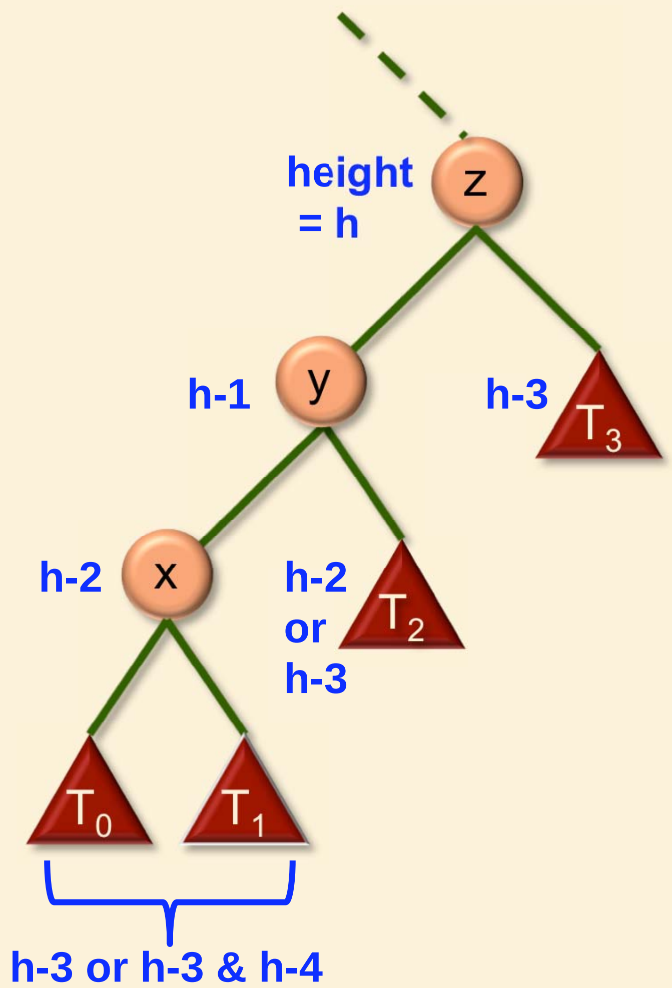
# Removal: Rebalancing Strategy

- Step 2: Repair
  - We again use **trinode restructuring**.
  - 3 nodes x, y and z are distinguished:
    - z = the parent of the high sibling
    - y = the high sibling
    - x = the high child of the high sibling (if children are equally high, keep chain linear)



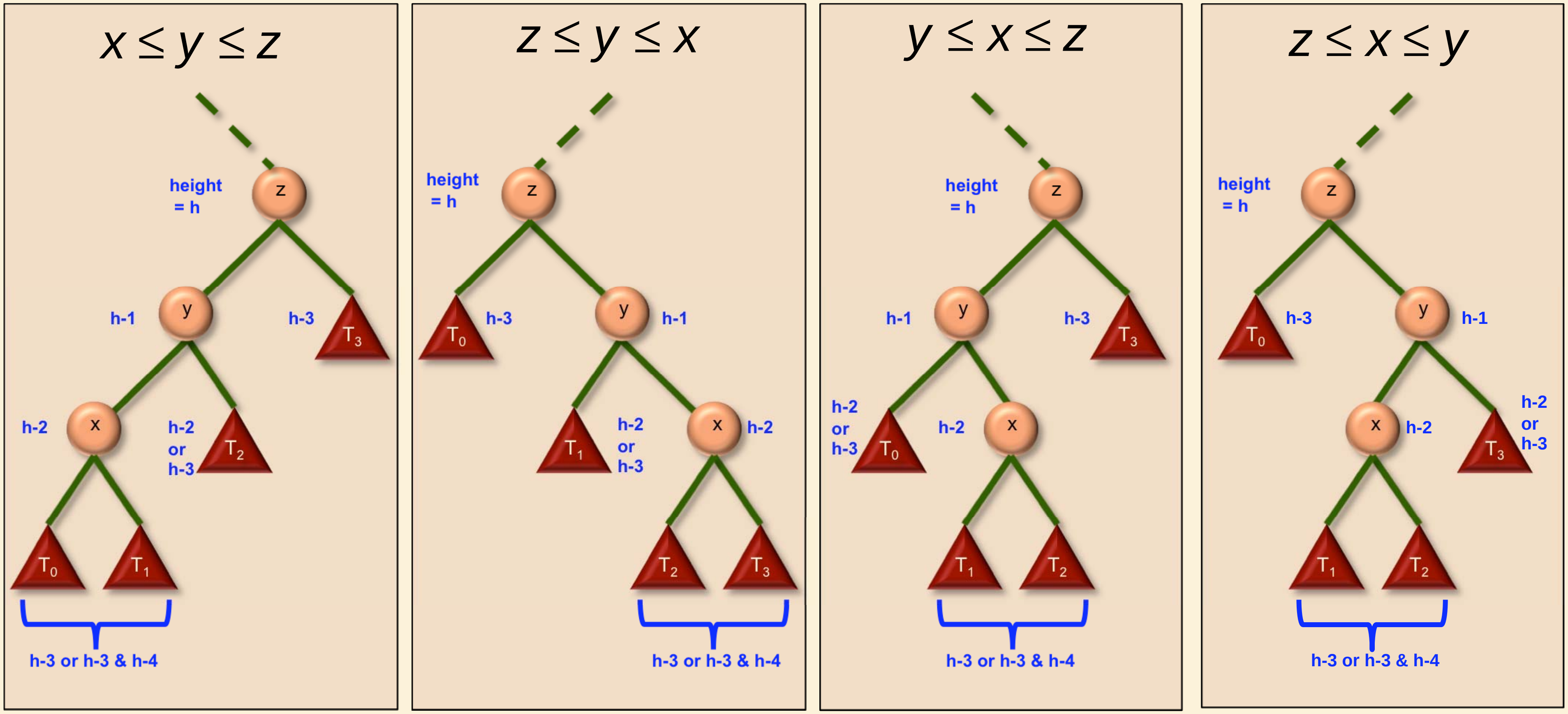
# Removal: Rebalancing Strategy

- Step 2: Repair
  - The idea is to rearrange these 3 nodes so that the middle value becomes the root and the other two becomes its children.
  - Thus the linear **grandparent – parent – child** structure becomes a triangular **parent – two children** structure.
  - Note that **z** must be either bigger than both **x** and **y** or smaller than both **x** and **y**.
  - Thus either **x** or **y** is made the root of this subtree, and **z** is lowered by 1.
  - Then the subtrees $T_0 - T_3$ are attached at the appropriate places.
  - Although the subtrees $T_0 - T_3$ can differ in height by up to 2, after restructuring, sibling subtrees will differ by at most 1.



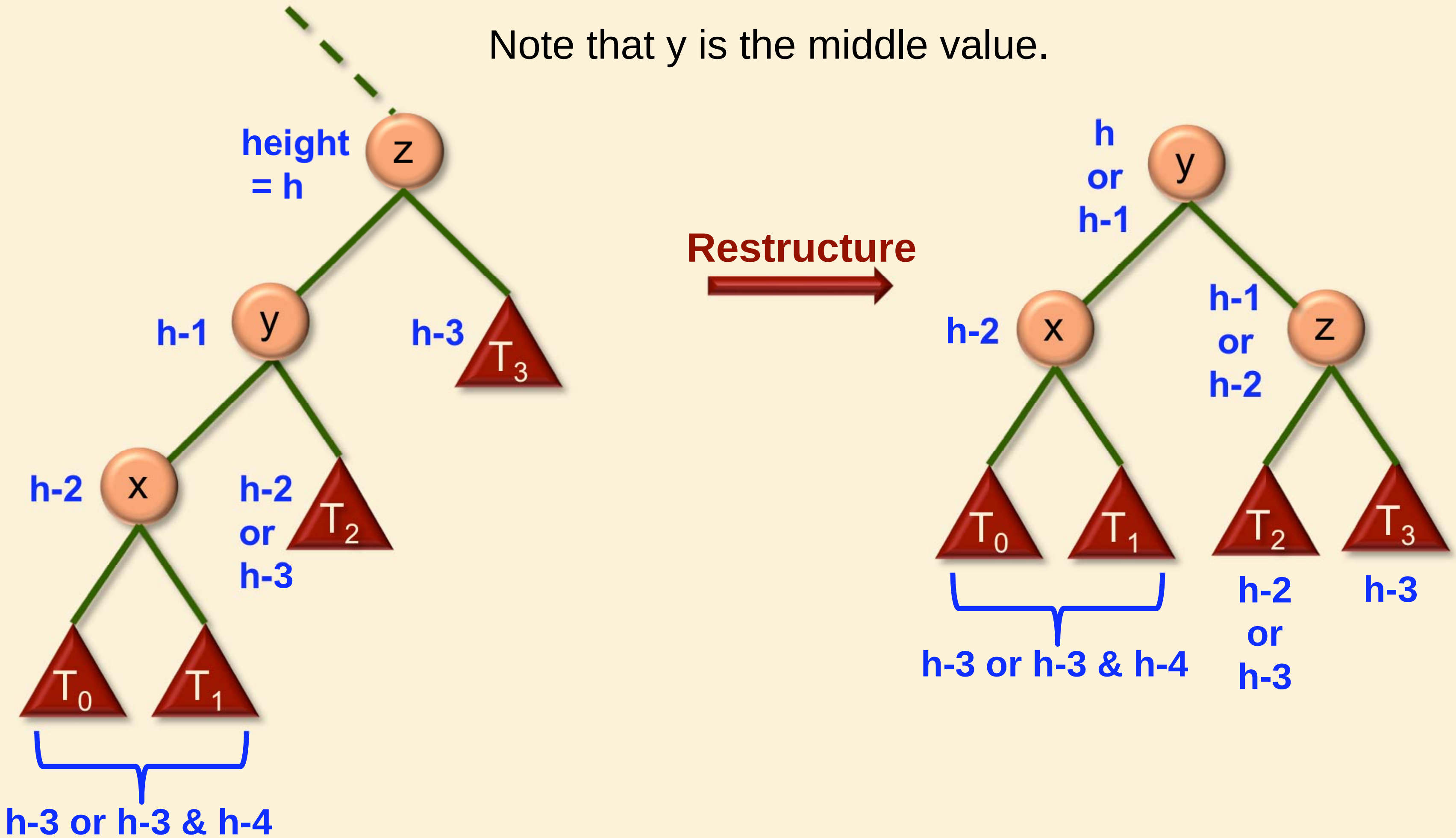
# Removal: Trinode Restructuring - 4 Cases

- There are 4 different possible relationships between the three nodes x, y and z before restructuring:

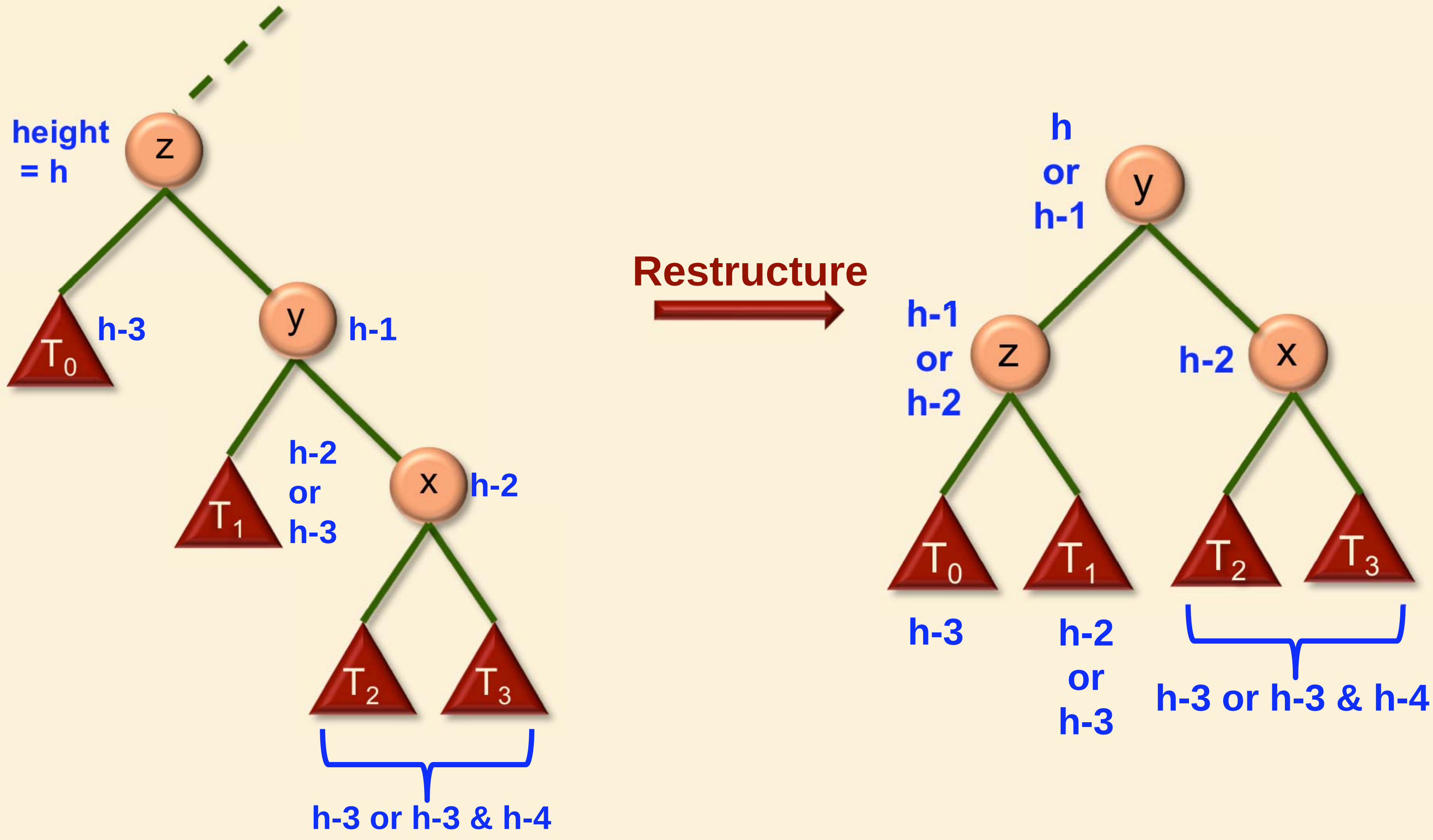


# Removal: Trinode Restructuring - Case 1

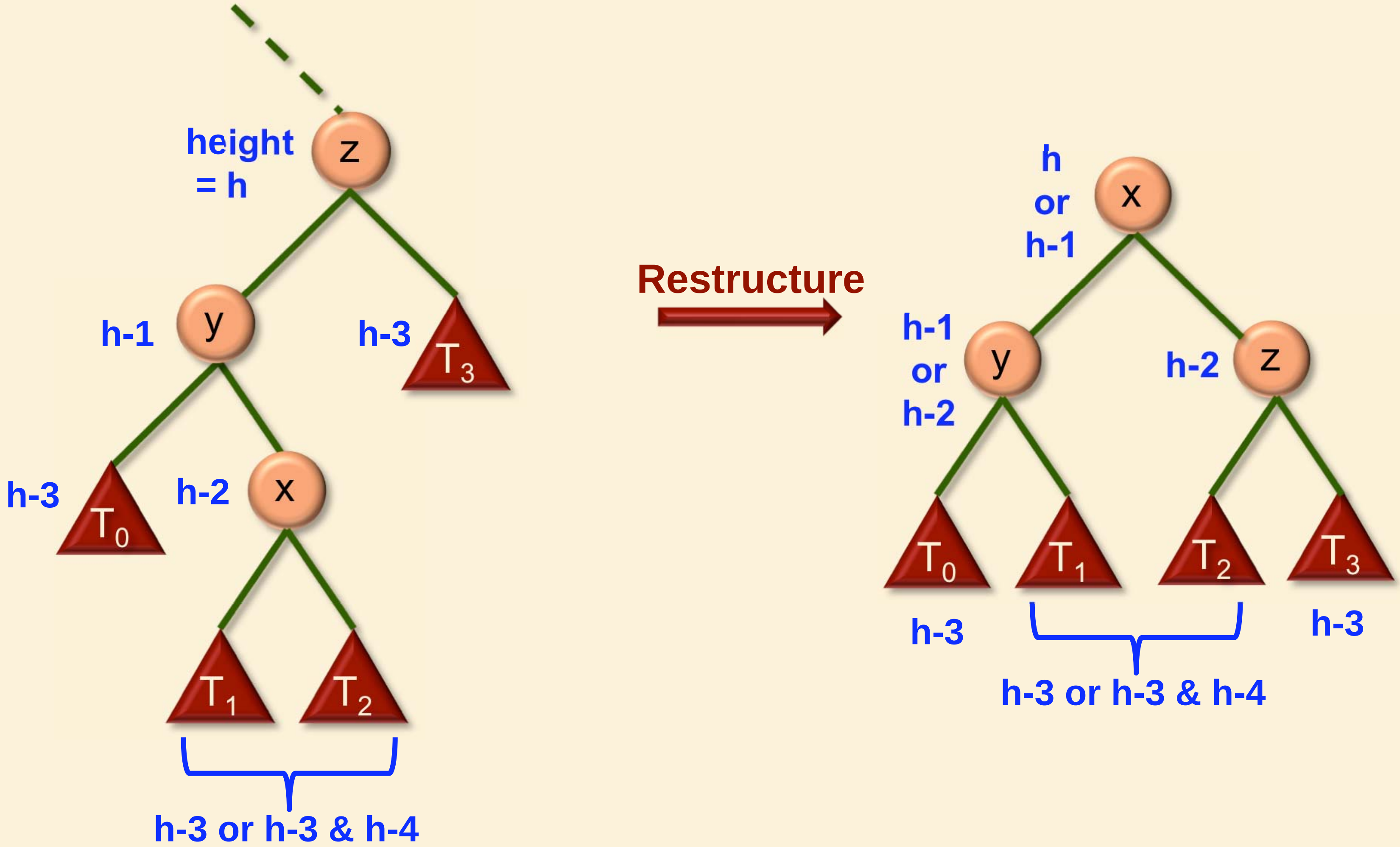
Note that y is the middle value.



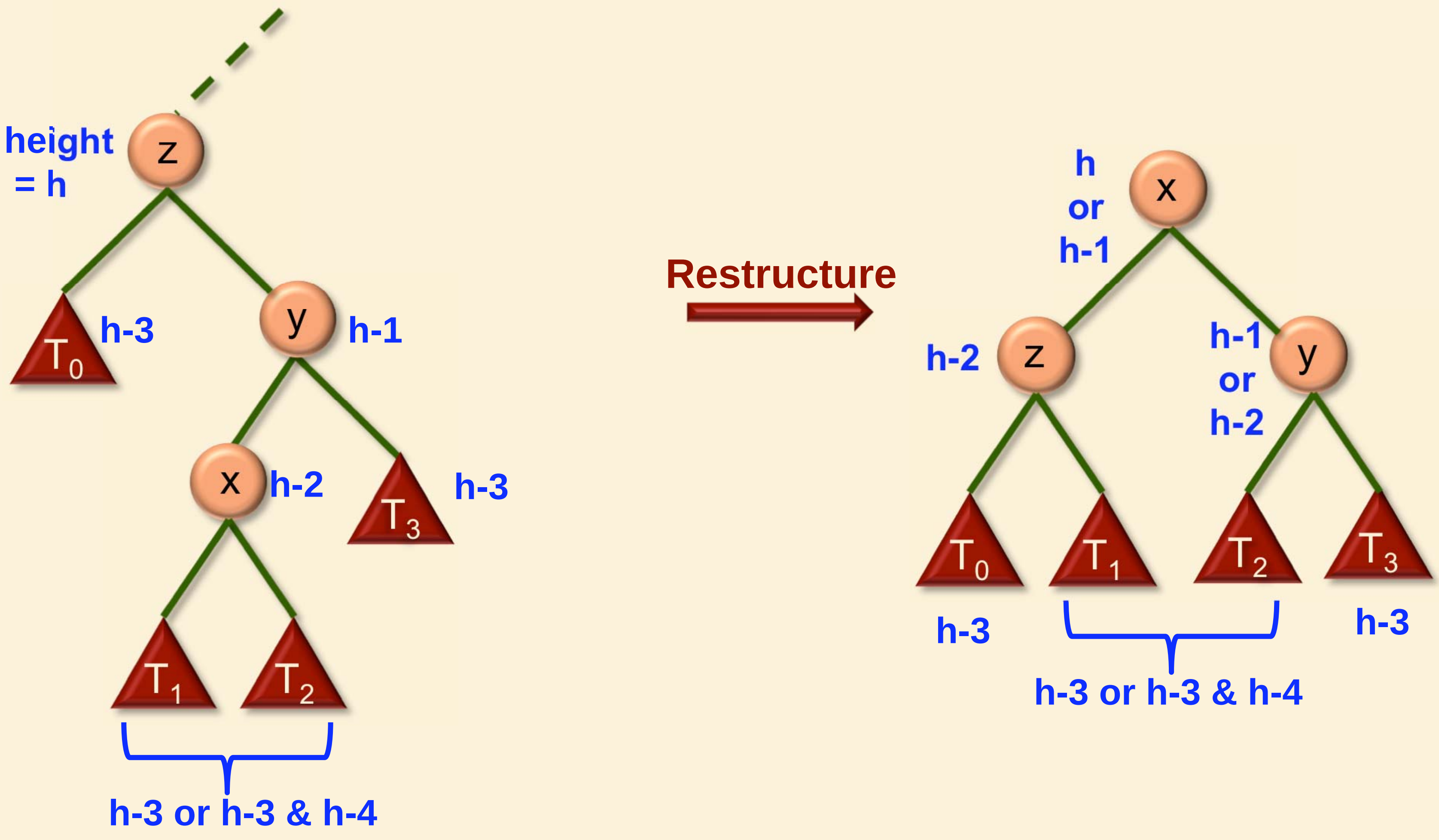
# Removal: Trinode Restructuring - Case 2



# Removal: Trinode Restructuring - Case 3



# Removal: Trinode Restructuring - Case 4



# Removal: Rebalancing Strategy

- Step 2: Repair
  - Unfortunately, trinode restructuring may reduce the height of the subtree, causing another imbalance further up the tree.
  - Thus this search and repair process must be repeated until we reach the root.

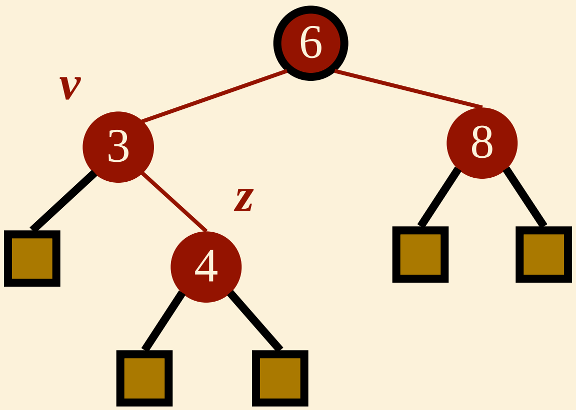
# Java Implementation of AVL Trees

- Please see text

# Running Times for AVL Trees

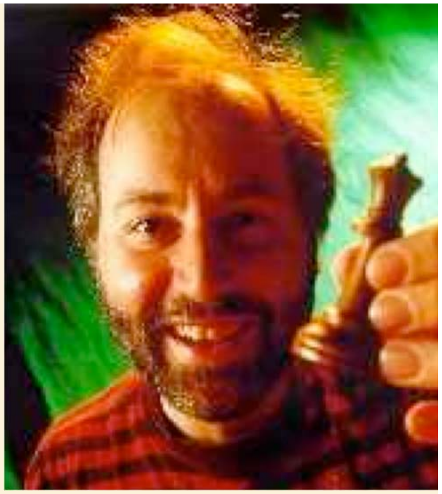
- a single restructure is O(1)
  - using a linked-structure binary tree
- find is O(log n)
  - height of tree is O(log n), no restructures needed
- insert is O(log n)
  - initial find is O(log n)
  - Restructuring is O(1)
- remove is O(log n)
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)

# Splay Trees



# Splay Trees

- Self-balancing BST
- Invented by Daniel Sleator and Bob Tarjan
- Allows quick access to recently accessed elements
- Bad: worst-case O(n)
- Good: average (amortized) case O(log n)
- Often perform better than other BSTs in practice



D. Sleator

R. Tarjan

# Splaying

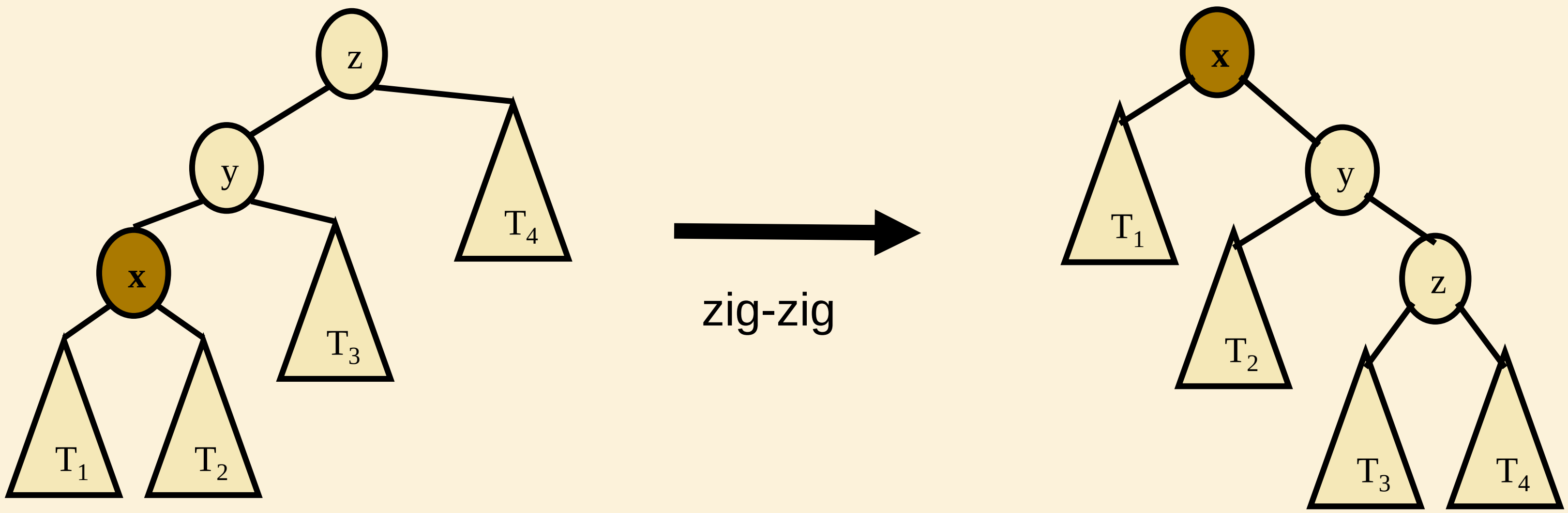
- Splaying is an operation performed on a node that iteratively moves the node to the root of the tree.
- In splay trees, each BST operation (find, insert, remove) is augmented with a splay operation.
- In this way, recently searched and inserted elements are near the top of the tree, for quick access.

# 3 Types of Splay Steps

- Each splay operation on a node consists of a sequence of splay steps.
- Each splay step moves the node up toward the root by 1 or 2 levels.
- There are 2 types of step:
  - Zig-Zig
  - Zig-Zag
  - Zig

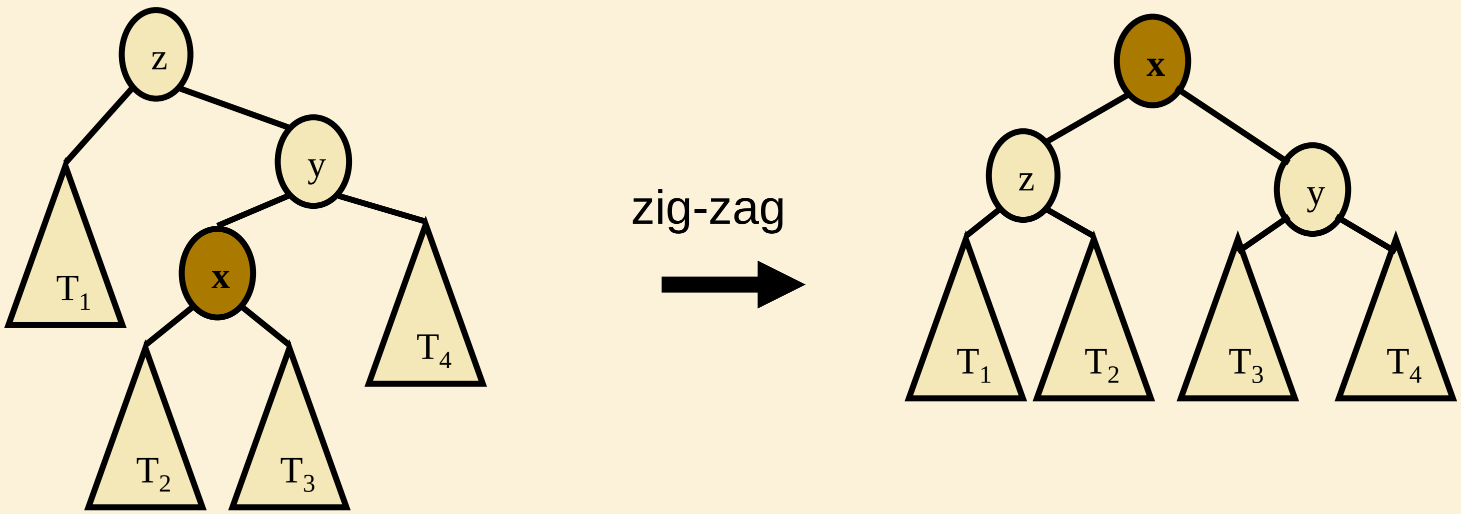
# Zig-Zig

- Performed when the node x forms a linear chain with its parent and grandparent.
  - i.e., right-right or left-left



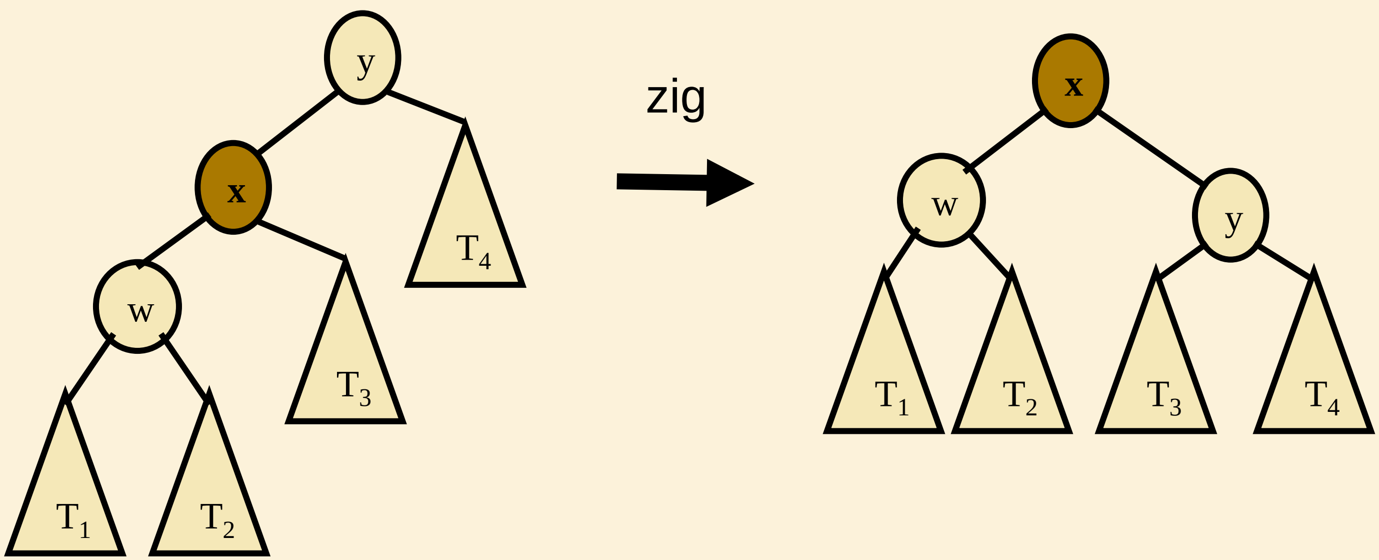
# Zig-Zag

- Performed when the node x forms a non-linear chain with its parent and grandparent
  - i.e., right-left or left-right



# Zig

- Performed when the node x has no grandparent
  - i.e., its parent is the root



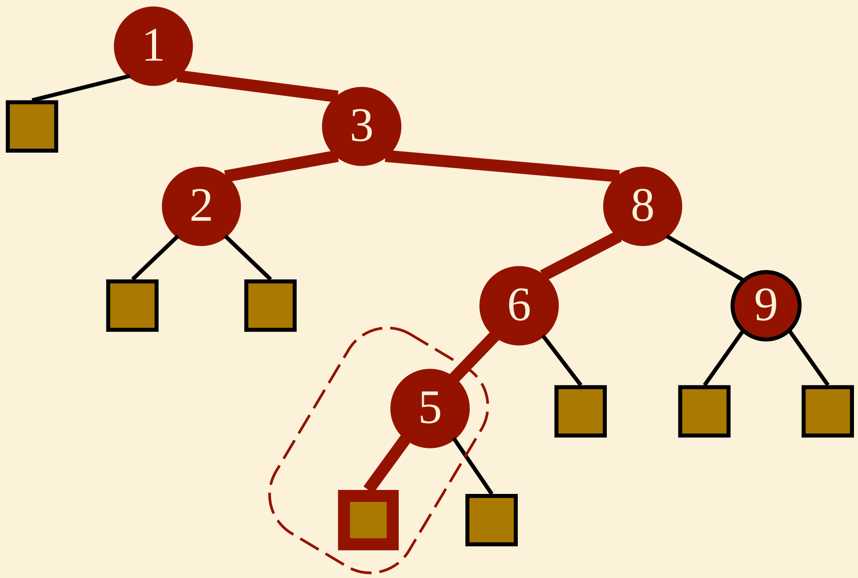
# Splay Trees & Ordered Dictionaries

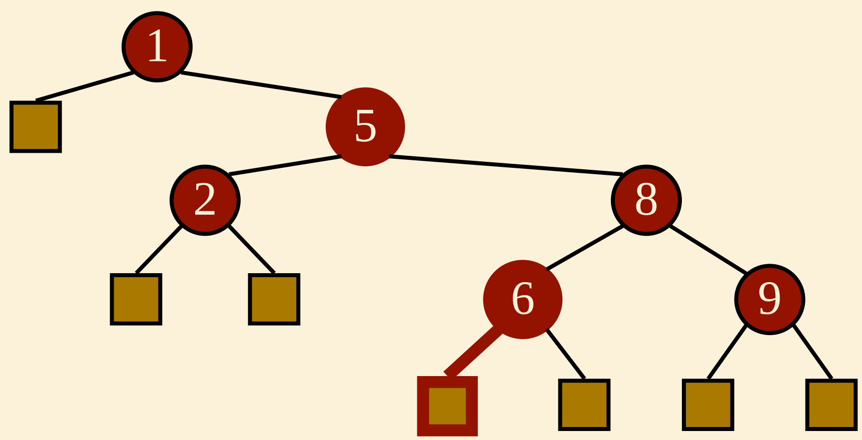
- which nodes are splayed after each operation?

| method | splay node |
|---|---|
| find(k) | if key found, use that node<br>if key not found, use parent of external node where search terminated |
| insert(k,v) | use the new node containing the entry inserted |
| remove(k) | use the parent of the internal node that was actually removed from the tree (the parent of the node that the removed item was swapped with) |

# Recall BST Deletion

- Now consider the case where the key ***k*** to be removed is stored at a node ***v*** whose children are both internal
  - we find the internal node ***w*** that follows ***v*** in an inorder traversal
  - we copy ***key*(*w*)** into node ***v***
  - we remove node ***w*** and its left child ***z*** (which must be a leaf) by means of operation removeExternal(***z***)
- Example: remove 3 – which node will be splayed?



# Note on Deletion

- The text (Goodrich, p. 445) uses a different convention for BST deletion in their splaying example
  - Instead of deleting the leftmost internal node of the right subtree, they delete the rightmost internal node of the left subtree.
  - We will stick with the convention of deleting the leftmost internal node of the right subtree (the node immediately following the element to be removed in an inorder traversal).

# Splay Tree Example

# Performance

- Worst-case is O(n)
  - Example:
    - Find all elements in sorted order
    - This will make the tree a left linear chain of height n, with the smallest element at the bottom
    - Subsequent search for the smallest element will be O(n)

# Performance

- Average-case is O(log n)
  - Proof uses amortized analysis
  - We will not cover this

# Other Forms of Search Trees

- (2, 4) Trees
  - These are multi-way search trees (not binary trees) in which internal nodes have between 2 and 4 children
  - Have the property that all external nodes have exactly the same depth.
  - Worst-case O(log n) operations
  - Somewhat complicated to implement
- Red-Black Trees
  - Binary search trees
  - Worst-case O(log n) operations
  - Somewhat easier to implement
  - Requires only O(1) structural changes per update